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The Relation between Behavior under Risk and over Time†

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The paper establishes a tight relation between nonstandard behaviors in the domains of risk and time, by considering a decision-maker with non-expected utility preferences who believes that only present consumption is certain while any future consumption is uncertain. We provide the first complete characterizations of the two-way relations between the certainty effect and present bias, and between the common ratio effect and temporal reversals. (*JEL* D11, D15, D81, D91)

This paper studies if and how behaviors in the domains of risk and time may be similar and related. This similarity is evident in the mutually mirroring mathematical models used for the analysis of behavior under risk and over time. The workhorse model of intertemporal choice, exponential discounting, evaluates the utility of a consumption stream by additively aggregating the utility of each consumption outcome, after exponentially weighting it by the associated time-delay. The canonical model for choice under risk, expected utility, *similarly* calculates the utility of a lottery by aggregating the utility of each possible outcome after weighting it by its respective probability. Further, these normative mathematical models contain similar descriptive inadequacies:

- Preferences are disproportionately sensitive to certainty (certainty effect) in the risk domain and to the present (present bias) in the time domain.
- Proportional changes in probabilities (common ratio effect) or the introduction of equal time delays (temporal reversals) affect the preferences between two alternatives disproportionately.[1]

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[1] Certainty effect and present bias are often taken as special cases of common ratio effect and temporal reversals, respectively.

Moreover, Keren and Roelofsma (1995) and Weber and Chapman (2005) provide experimental evidence that introducing explicit risk to immediate rewards almost eliminates present bias, while introducing delay to sure outcomes almost eliminates the certainty effect. These parallels are well accepted in the literature (Green and Myerson 2004, Chapman and Weber 2006), and there is an implicit understanding that the existence of such mirroring behaviors is not a mere coincidence but points to a common fundamental property of decision-making that manifests itself across domains of behavior (Prelec and Loewenstein 1991, Baucells and Heukamp 2012). For example, a delayed reward or consumption could be inherently risky as there might be events between the current date and the promised date that interfere in the process of acquiring the reward (Halevy 2008). This would explain why risk preferences could manifest in intertemporal choice patterns. Rachlin et al. (1986) and Rachlin, Brown, and Cross (2000) suggested the opposite direction; if the utility of probabilistic rewards were calculated using mean waiting time before a successful draw of the corresponding reward, then time preferences could be used to derive preferences over the probabilistic rewards. The current paper formalizes these intuitions and provides a two-way characterization of how prominent behavioral traits from the domains of risk and time could be related. We prove our results in two commonly used decision domains: one where the decision-maker (DM) is choosing between temporal rewards ($X \times \mathbb{R}_+$) with the set of time periods being the set of all nonnegative numbers and the consumption set is a subset X of $\mathbb{R}_+$ and another where the DM is choosing from the space of consumption streams ($X^{\mathbb{N}}$, where $\mathbb{N}$ are nonnegative integers).

Section I introduces some of the basic concepts from Halevy (2008). Sections II and III provide the relevant definitions from the domains of risk and intertemporal behavior, respectively. In Section IV, we state and prove our main results. The counterexample that shows the incompleteness of characterization results in the previous literature (described in Table 2) is included in online Appendix A.

I. Background

The idea that Diminishing Impatience (present bias or quasi-hyperbolic discounting) may be related to the certainty of the present and the risk associated with future rewards, was formalized by Halevy (2008) in the domain of preferences over consumption streams. In this model, every consumption path $\mathbf{c} = (c_0, c_1, c_2, \ldots)$ is subject to some constant hazard rate r of termination, with only the first period of consumption (at $t = 0$) being certain. The DM chooses as if he has the following utility function over consumption paths: he calculates present discounted utility for every possible length of the path (all periods before termination of consumption). The distribution over present discounted utilities is then evaluated using Rank Dependent Utility (RDU, see Remark 1) with probability weighting function $g(\cdot)$. The DM's preferences over consumption streams, for any particular $r \in (0,1)$, is represented by

$$(1) \qquad U(\mathbf{c}, r) = \sum_{t=0}^{\infty} g\big((1-r)^t\big)\,\delta^t u(c_t),$$

where δ is a constant pure time preference parameter and $u(\cdot)$ is the felicity function. Given the representation (1), the composite discount function at period t is

$$D(t) = \delta^t g\big((1-r)^t\big). \tag{2}$$

The DM's *impatience* at time t is the ratio of her composite discount functions at periods t and $t+1$. Halevy (2008) defines Diminishing Impatience as the property of "impatience being maximized at $t = 0$."

II. Risky Behavior

In this section, we consider a monotone (with respect to first-order stochastic dominance) risk preference $\succsim^r$ on the set of binary lotteries:

$$\Delta = \{(x,p)\,|\,x \in X \text{ and } p \in [0,1]\},$$

where X is a non-degenerate closed interval in $\mathbb{R}_+$ including 0, and (x,p) is a lottery that pays x with probability p and 0 with probability $1-p$. We denote the symmetric and the asymmetric parts by $\sim^r$ and $\succ^r$, respectively.

Suppose the subject chooses between a safer option which pays x with probability η, and a riskier option which provides a larger gain y with probability $\eta\mu$, where $\mu < 1$. A subject who exhibits the common ratio effect switches his choice, as η falls, from the safer to the riskier option. Formally, we have the following.

DEFINITION 1: $\succsim^r$ *is said to exhibit*

(i) Strict Common Ratio Effect[2] *if, for any* $x,y \in X$, $\mu,\tilde{\eta} \in (0,1]$ *such that* $x < y$ *and* $(x,\tilde{\eta}) \sim^r (y,\tilde{\eta}\mu)$:

$$(x,\eta) \prec^r (y,\eta\mu) \quad \textit{for all } \eta \in (0,\tilde{\eta}), \tag{3a}$$

$$(x,\eta) \succ^r (y,\eta\mu) \quad \textit{for all } \eta \in (\tilde{\eta},1]. \tag{3b}$$

(ii) Weak Common Ratio Effect *if the conclusion in equation* (3) *holds with weak preferences and there exist some* x, y *and* $\mu,\tilde{\eta},\eta$ *such that the conclusion in equation* (3) *holds* (*with strict preferences*).

The general definition provided by Machina (1982) is equivalent to the above definition within the set of binary lotteries. Certainty effect is a special case of the common ratio effect, when $\tilde{\eta} = 1$.

DEFINITION 2: $\succsim^r$ *is said to exhibit*

[2] Under the standard axioms of monotonicity and continuity, for any $x,y \in X$ and $\tilde{\eta} \in [0,1]$, there exists μ such that $(x,\tilde{\eta}) \sim^r (y,\tilde{\eta}\mu)$. So the condition cannot be satisfied in a trivial way.

(i) Strict Certainty Effect *if, for any* $x, y \in X$ *and* $\mu \in [0,1)$ *such that* $x < y$,

$$(x,1) \sim^r (y,\mu) \Rightarrow (x,\eta) \prec^r (y,\eta\mu) \quad \textit{for all } \eta \in (0,1). \tag{4}$$

(ii) Weak Certainty Effect *if the conclusion of equation* (4) *holds with weak preferences and there exist some* x, y *and* μ, η *such that the conclusion of equation* (4) *holds* (*with strict preferences*).

Finally, in the set of binary lotteries, the Independence Axiom reduces to the following definition.

DEFINITION 3: $\succsim^r$ *satisfies the* Independence Axiom *if, for any* $x, y \in X$ *and* $\mu, \eta, \eta' \in (0,1]$,

$$(x,\eta) \succsim^r (y,\eta\mu) \Leftrightarrow (x,\eta') \succsim^r (y,\eta'\mu).$$

REMARK 1: *Assume the DM's preferences over binary lotteries are represented by Rank Dependent Utility* (*RDU*), $U(x,p) = u(x)g(p)$ *where* $u(\cdot)$ *is a real-valued increasing function on X and* $g:[0,1] \rightarrow [0,1]$ *is a probability weighting function and for any* $\alpha \in \mathbb{R}_+$ *there exist* $x, y \in X$ *such that* $\alpha = u(y)/u(x)$. *Then, the DM exhibits*

(i) Strict Common Ratio Effect *if and only if for all* $p, q \in (0,1)$ *and* $\ell \in (0,1]$,

$$\frac{g(\ell)}{g(p\ell)} > \frac{g(q\ell)}{g(pq\ell)}. \tag{5}$$

(ii) Weak Common Ratio Effect *if and only if equation* (5) *holds with weak inequality and there exist* p, q, ℓ *for which equation* (5) *holds* (*with strict inequality*).

(iii) Strict Certainty Effect *if and only if* $p, q \in (0,1)$,

$$g(pq) > g(p)g(q). \tag{6}$$

(iv) Weak Certainty Effect *if and only if equation* (6) *holds with weak inequality and there exist* p, q *for which equation* (6) *holds* (*with strict inequality*).

PROOF:

We show (i), (ii)–(iv) follow similarly. Assume equation (3) to show equation (5). Fix $p, q \in (0,1)$, $\ell \in (0,1]$. By assumption, we can find x, y such that $u(y)/u(x) = g(\ell)/g(\ell p)$. By equation (3),

$$\frac{g(\ell)}{g(\ell p)} = \frac{u(y)}{u(x)} > \frac{g(\eta)}{g(\eta p)} \quad \text{for all } \eta < \ell.$$

Let $\eta = q\ell$. Then $\eta < \ell$ and $g(\ell)/g(\ell p) > g(q\ell)/g(pq\ell)$.

To show the converse, fix $x, y, \mu, \tilde{\eta}$ such that $(x, \tilde{\eta}) \sim^r (y, \tilde{\eta}\mu)$. Then $g(\tilde{\eta})/g(\tilde{\eta}\mu) = u(y)/u(x)$. For equation (3a), fix $\eta < \tilde{\eta}$. Let $\ell := \tilde{\eta}$, $p := \mu$, $q := \eta/\tilde{\eta}$. Then $q < 1$ and

$$\frac{g(\ell)}{g(p\ell)} > \frac{g(q\ell)}{g(pq\ell)} \Rightarrow \frac{u(y)}{u(x)} = \frac{g(\tilde{\eta})}{g(\tilde{\eta}\mu)} > \frac{g(\eta)}{g(\eta\mu)} \Rightarrow u(x)g(\eta) < u(y)g(\eta\mu),$$

or $(x, \eta) \prec^r (y, \eta\mu)$. To show equation (3b) fix η such that $\eta > \tilde{\eta}$. The result follows by letting $q := \tilde{\eta}/\eta$, $p := \mu$, $\ell := \eta$. Then $q < 1$ and

$$\frac{g(\ell)}{g(p\ell)} > \frac{g(q\ell)}{g(pq\ell)} \Rightarrow \frac{g(\eta)}{g(\eta\mu)} > \frac{g(\tilde{\eta})}{g(\tilde{\eta}\mu)} = \frac{u(y)}{u(x)} \Rightarrow u(x)g(\eta) > u(y)g(\eta\mu),$$

or $(x, \eta) \succ^r (y, \eta\mu)$. ∎

III. Intertemporal Behavior

In this section, we define preferences that subsume the classes of exponential, hyperbolic and quasi-hyperbolic discounting. We denote the set of time periods by T. We consider two cases: when preferences are defined over temporal rewards ($X \times T$ when $T = \mathbb{R}_+$), and over consumption streams (X^T when $T = \mathbb{N}$).

A. *Temporal Rewards in Continuous Time*

Preferences are defined over pairs of prospects, where each prospect consists of a reward $x \in X$ at time $t \in T = \mathbb{R}_+$. For each $d \in T$, we denote the set of temporal rewards, paid after time d, by $X(d) = \{[x,t] \,|\, x \in X \text{ and } t \in T \text{ such that } t \geq d\}$. The DM's time-indexed preferences are given by $\{\succsim_d\}_{d \in T}$, where $\succsim_d$ is a binary relation on $X(d)$ for each decision time $d \in T$.[3]

Hyperbolic discounting implies the following pattern of dynamic choice: the DM chooses a later larger reward over an earlier smaller reward, but reverses his choice as both reward dates approach the decision date.[4] Temporal Reversal formalizes this behavioral pattern as follows.[5]

DEFINITION 4: $\{\succsim_d\}_{d \in T}$ *is said to exhibit*

(i) Temporal Reversal *if, for any* $x, y \in X$ *and* $\tilde{d}, t, s \in T$ *such that* $x < y$, $\tilde{d} \leq t < s$, *and* $[x,t] \sim_{\tilde{d}} [y,s]$:

(7a) $$[x,t] \prec_d [y,s] \quad \textit{for all } d \textit{ such that } d < \tilde{d},$$

[3] For each $d \in T$, we denote the symmetric and the asymmetric parts of $\succsim_d$ by $\sim_d$ and $\succ_d$, respectively.

[4] In the following three definitions of time preferences, we focus on rewards that provide positive utility, for simplicity. For the case of rewards that could provide disutility instead (effort, for example), present bias appears as procrastination and is defined in the same way by switching strict preference from $\succ$ to $\prec$, and vice versa. O'Donoghue and Rabin (1999) offer examples of procrastination and Halevy (2008) discusses how to incorporate into the current framework using the reflection effect.

[5] Similar to Proposition 1 in Dasgupta and Maskin (2005).

(7b) $$[x,t] \succ_d [y,s] \quad \textit{for all } d \textit{ such that } \tilde{d} < d < t.$$

(ii) Present-Biased Temporal Reversal *if, for any* $x, y \in X$ *and* $t, s \in T$ *such that* $x < y$ *and* $t < s$,

$$[x,t] \sim_t [y,s] \Rightarrow [x,t] \prec_d [y,s] \quad \textit{for all } d < t.$$

(iii) Temporally Unbiased *if, for any* $x, y \in X$ *and* $d, d', s, t \in \mathbb{R}_+$,

$$[x,t] \succsim_d [y,s] \Leftrightarrow [x,t] \succsim_{d'} [y,s].$$

Present-Biased Temporal Reversal is a special case of Temporal Reversal when $\tilde{d} = t$. Temporally Unbiased preferences are time consistent.

B. *Consumption Streams in Discrete Time*

We now consider consumption streams in discrete time (i.e., $T = \mathbb{N}$). In this case, temporal behavior is usually characterized by properties of the discount-function, and under time-invariance it depends only on the distance between the evaluation time and consumption time. Let $D(\cdot)$ be the DM's discount function, so the utility of consuming x after τ periods is $D(\tau)u(x)$, where u is a real-valued function on X. The function D exhibits *hyperbolic discounting* if $D(\tau) = 1/(1+\rho\tau)$ for some $\rho > 0$; *quasi-hyperbolic discounting* if $D(0) = 1$ and $D(\tau) = \beta\delta^{\tau}$ for some $\delta \in (0,1]$ and $\beta < 1$ for all $\tau \geq 1$.

The DM's (one period) impatience at t is $D(t)/D(t+1)$. DM's (k period) impatience at t is $D(t)/D(t+k)$. In Table 1 we define the notions of temporal behavior for streams ($X^{\mathbb{N}}$).

PROPOSITION 1:

(i) *Delay Independent Diminishing Impatience implies Diminishing Impatience (but not the converse).*

(ii) *Strongly Diminishing Impatience and Delay Independent Strongly Diminishing Impatience are equivalent.*

PROOF:

(i) Take $k = 1$. An implication of the counterexample provided in online Appendix A is that Diminishing Impatience does not imply Delay Independent Diminishing Impatience.

(ii) Delay Independent Strongly Diminishing Impatience trivially implies Strongly Diminishing Impatience. To show the converse fix k, t', t such that $t' > t$ to show that $D(t)/D(t+k) > D(t')/D(t'+k)$. Notice that

$$\frac{D(t)}{D(t+k)} = \underbrace{\frac{D(t)}{D(t+1)}\frac{D(t+1)}{D(t+2)}\cdots\frac{D(t+k-1)}{D(t+k)}}_{k \text{ terms}} = \prod_{d=0}^{k-1}\frac{D(t+d)}{D(t+d+1)},$$

TABLE 1—NOTIONS OF TEMPORAL BEHAVIOR FOR CONSUMPTION STREAMS ($X^{\mathbb{N}}$)

	Definition
Diminishing Impatience	$\frac{D(0)}{D(1)} > \frac{D(t)}{D(t+1)} \quad \forall t \in \mathbb{N}_+$
Delay Independent Diminishing Impatience	$\frac{D(0)}{D(k)} > \frac{D(t)}{D(t+k)} \quad \forall k, t \in \mathbb{N}_+$
Strongly Diminishing Impatience	$\frac{D(t)}{D(t+1)} > \frac{D(t')}{D(t'+1)} \quad \forall t, t' \in \mathbb{N}$ with $t < t'$
Delay Independent Strongly Diminishing Impatience	$\frac{D(t)}{D(t+k)} > \frac{D(t')}{D(t'+k)} \quad \forall t, t' \in \mathbb{N}, k \in \mathbb{N}_+$ with $t < t'$

$$\frac{D(t')}{D(t'+k)} = \underbrace{\frac{D(t')}{D(t'+1)} \frac{D(t'+1)}{D(t'+2)} \cdots \frac{D(t'+k-1)}{D(t'+k)}}_{k \text{ terms}} = \prod_{d=0}^{k-1} \frac{D(t'+d)}{D(t'+d+1)}.$$

Since $t' > t$, by Strongly Diminishing Impatience, we have for each $d \in \{0, \ldots, k-1\}$, $D(t+d)/D(t+d+1) > D(t'+d)/D(t'+d+1)$. Hence, $D(t)/D(t+k) > D(t')/D(t'+k)$. ■

Diminishing Impatience and Strongly Diminishing Impatience have been proposed by Halevy (2008). Delay Independent Diminishing Impatience and Delay Independent Strongly Diminishing Impatience are new properties motivated by the hyperbolic discounting and the quasi-hyperbolic discounting models (see Proposition 2), and they describe the failure of stationarity independently of the delay under consideration. Delay Independent Diminishing Impatience requires impatience to diminish for all possible delays ($k \geq 1$), hence is a strengthening of Diminishing Impatience.

PROPOSITION 2:

(*i*) *Quasi-hyperbolic discounting satisfies Delay Independent Diminishing Impatience but not Delay Independent Strongly Diminishing Impatience.*

(*ii*) *Hyperbolic discounting satisfies Delay Independent Strongly Diminishing Impatience (and hence Strongly Diminishing Impatience and Delay Independent Diminishing Impatience).*

PROOF:

(i) For $t' > t > 0$,

$$\frac{D(0)}{D(k)} = \frac{1}{\beta\delta^k} > \frac{1}{\delta^k} = \frac{D(t)}{D(t+k)} = \frac{D(t')}{D(t'+k)}.$$

(ii) For arbitrary k, and $t' > t \geq 0$,

$$\frac{D(t)}{D(t+k)} = 1 + \frac{\rho k}{1+\rho t} > 1 + \frac{\rho k}{1+\rho t'} = \frac{D(t')}{D(t'+k)}. \blacksquare$$

Finally, we elaborate on the relationship between the definitions for temporal rewards in continuous time (Definition 4) and the definitions for consumption streams in discrete time (definitions in Table 1). To achieve this we consider the intersection of their domains, which is temporal rewards in discrete time.

PROPOSITION 3: *Suppose that* $T = \mathbb{N}$ *and there exist* $D : T \to [0,1]$ *and* $u : X \to \mathbb{R}_+$ *such that for each* $d \in T$, $\succsim_d$ *is represented by* $U_d([x,t]) = D(t-d)u(x)$ *and for any* $\alpha \in \mathbb{R}_+$ *there exist* $x, y \in X$ *such that* $\alpha = u(x)/u(y)$. *Then, the following results hold:*

(*i*) *D exhibits Strongly Diminishing Impatience* (*and hence Delay Independent Strongly Diminishing Impatience*) *if and only if* $\{\succsim_d\}_{d \in T}$ *exhibits Temporal Reversal.*

(*ii*) *D exhibits Delay Independent Diminishing Impatience if and only if* $\{\succsim_d\}_{d \in T}$ *exhibits Present-Biased Temporal Reversal.*

PROOF:

For this proof, we will use the result that Strongly Diminishing Impatience and Delay Independent Strongly Diminishing Impatience are equivalent. To show (i), suppose that $\{\succsim_d\}$ exhibit Temporal Reversals and show that D exhibits Delay Independent Strongly Diminishing Impatience, i.e, for an arbitrary nonnegative integer $\tau' < \tau$ we will show that

$$\frac{D(\tau')}{D(\tau'+k)} > \frac{D(\tau)}{D(\tau+k)}.$$

Choose $x, y \in X$ such that $D(\tau')/D(\tau'+k) = u(y)/u(x)$. Hence, $[x,\tau] \sim_{\tau-\tau'} [y, \tau+k]$. Then by Temporal Reversals we have $[x,\tau] \prec_0 [y,\tau+k]$. This means that $D(\tau)u(x) < D(\tau+k)u(y)$. It follows that, $D(\tau')/D(\tau'+k) = u(y)/u(x) > D(\tau)/D(\tau+k)$. Since the choice of τ is arbitrary, this means that D exhibits Strongly Diminishing Impatience.

For the converse direction, suppose that there exist $x, y \in X$ and $\tilde{d}, t, s \in T$ such that $[x,t] \sim_{\tilde{d}} [y,s]$ and $\tilde{d} \leq t \leq s$. Choose $d, d' \in T$ such that $d < \tilde{d}$ and $d' > \tilde{d}$ to show $[x,t] \prec_d [y,s]$ and $[x,t] \succ_{d'} [y,s]$.

Since $[x,t] \sim_{\tilde{d}} [y,s]$, by definition, $D(t-\tilde{d})u(x) = D(s-\tilde{d})u(y)$, so that

$$\frac{D(t-\tilde{d})}{D(s-\tilde{d})} = \frac{u(y)}{u(x)}. \tag{8}$$

Since $d < \tilde{d} < d'$, it follows from Delay Independent Strongly Diminishing Impatience that

$$\frac{D(t-d)}{D(s-d)} < \frac{D(t-\tilde{d})}{D(s-\tilde{d})} < \frac{D(t-d')}{D(s-d')}.$$

By equation (8), we get $D(t-d)u(x) < D(s-d)u(y)$ and $D(t-d')u(x) > D(s-d')u(y)$. Hence, $[x,t] \prec_d [y,s]$ and $[x,t] \succ_{d'} [y,s]$.

To show (ii) take $\tau' = 0$ in the forward direction to get Delay Independent Diminishing Impatience from Present-Biased Temporal Reversal. Use $t = \tilde{d}$ and consider $d < \tilde{d}$ for the converse. ∎

IV. Results

A. *Results for Temporal Rewards in Continuous Time*

In this subsection, we assume $T = \mathbb{R}_+$. We assume that the DM's temporal and risk preferences are connected in the following way: the individual discounts a future reward because he is uncertain whether he can consume it. We model this uncertainty through a stopping process that determines the last period until which rewards are available. Let $p(t) = e^{-rt}$ be the probability that the DM may collect a reward at time t, where $r \in (0,1)$ is the hazard rate.

At time d, such that $0 \leq d \leq t$, the DM updates the probability that a reward is available at time t according to the conditional probability: $p(t|d) = p(t)/p(d) = e^{-r(t-d)}$. Therefore, at time d he prefers receiving the temporal reward $[x,t]$, to another reward $[y,s]$ if and only if his risk preferences rank the binary lottery $(x,p(t|d))$ (which pays x with probability $p(t|d)$) over the lottery $(y,p(s|d))$. Thus, the DM's time preferences $\{\succsim_d\}_{d\in T}$ for each decision time $d \in T$ and risk preferences $\succsim^r$ are related as follows.

ASSUMPTION 1: *For all $d \in T$ and $[x,t],[y,s] \in X(d)$,*

$$[x,t] \succsim_d [y,s] \Leftrightarrow (x,p(t|d)) \succsim^r (y,p(s|d)). \tag{9}$$

Our formulation of $p(t)$ implies that immediate rewards are certain, but as the promised date for future rewards becomes increasingly distant, the probability of receiving the reward exponentially decreases to zero. The additional property of $p(t|d) = p(t|s)p(s|d)$ would be very useful in the results that follow.

THEOREM 1: *Under Assumption* 1,

(*i*) $\succsim^r$ *exhibits Strict Common Ratio Effect if and only if* $\{\succsim_d\}_{d\in T}$ *exhibit Temporal Reversal.*

(*ii*) $\succsim^r$ *exhibits Strict Certainty Effect if and only if* $\{\succsim_d\}_{d\in T}$ *exhibit Present-Biased Temporal Reversal.*

(*iii*) $\succsim^r$ *satisfies the Independence Axiom if and only if* $\{\succsim_d\}_{d\in T}$ *is Temporally Unbiased.*

PROOF:

To prove (i), suppose that $\succsim^r$ exhibits (3a). Choose any $x,y \in X$ and $\tilde{d},t,s \in T$ such that $[x,t] \sim_{\tilde{d}} [y,s]$ and $\tilde{d} \leq t \leq s$. Then by definition, $(x,p(t|\tilde{d})) \sim^r (y,p(s|\tilde{d})) = (y,p(s|t)p(t|\tilde{d}))$. Fix $d < \tilde{d}$ to show $[x,t] \prec_d [y,s]$. Since p is strictly decreasing, $p(d) > p(\tilde{d})$, so that $p(t|d) < p(t|\tilde{d})$. So (3a) $\Rightarrow$ $(x, p(t|d)) \prec^r (y, p(s|t)p(t|d)) = (y, p(s|d))$. Then by definition, $[x,t] \prec_d [y,s]$. So (7a) holds. In the same way, we can show that equation (3b) implies Temporal Reversal (7b).

To show the converse, suppose that $\{\succsim_d\}$ exhibits (7a). Choose any $x, y \in X$ and $\mu, \tilde{\eta} \in [0,1]$ such that $(x, \tilde{\eta}) \sim^r (y, \tilde{\eta}\mu)$. Fix $\eta \in (0, \tilde{\eta})$ to show $(x, \eta) \prec^r (y, \eta\mu)$. Since p is a strictly decreasing bijection to $[0,1]$, there exist t and $\tilde{d}$ such that $t \geq \tilde{d} > 0$ and $p(t) = \eta$ and $p(\tilde{d}) = \eta/\tilde{\eta}$. Then, $p(t|\tilde{d}) = \tilde{\eta}$. Also, there exists s such that $s \geq t$ and $p(s) = \mu\eta$. Then, $p(s|t) = \mu$. Hence, $(x, p(t|\tilde{d})) \sim^r (y, p(s|t)p(t|\tilde{d})) = (y, p(s|\tilde{d}))$, so that $[x,t] \sim_{\tilde{d}} [y,s]$, by definition. Therefore, by (7a), $[x,t] \prec_0 [y,s]$. So the definition shows that $(x,\eta) = (x, p(t)) \prec^r (y, p(s)) = (y, \eta\mu)$. So (3a) holds. Similarly, we can show that equation (7b) implies equation (3b).

The proof of part (ii) is very similar to part (i) and is hence omitted.

To show (iii), suppose that $\succsim^r$ satisfies the Independence Axiom. Choose any $x, y \in X$ and $t, s, d, d' \in T$ such that $[x,t] \succsim_d [y,s]$ to show $[x,t] \succsim_{d'} [y,s]$. Since $[x,t] \succsim_d [y,s]$, by definition $(x, p(t|d)) \succsim^r (y, p(s|d))$. Consider the case in which $d > d'$. By the Independence Axiom, $(x, p(t|d')) = (x, p(t|d)p(d|d')) \succsim^r (y, p(s|d)p(d|d')) = (y, p(s|d'))$. By the definition, $[x,t] \succsim_{d'} [y,s]$. The proof for the other case in which $d' > d$ is similar.[6]

To show the converse, suppose that $\{\succsim_d\}$ is Temporally Unbiased. Choose any $x, y \in X$ and $\mu, \eta, \eta' \in [0,1]$ such that $(x,\eta) \succsim^r (y, \eta\mu)$ to show $(x, \eta') \succsim^r (y, \eta'\mu)$. Consider the case where $\eta' > \eta$. Since p is strictly decreasing and bijection to $[0,1]$, there exist $t, s, d \in T$ such that $s \geq t$, $p(t) = \eta$, $p(s) = \eta\mu$, and $p(d) = \eta/\eta'$. Then, $p(t|d) = \eta'$ and $p(s|d) = \eta'\mu$. Since $\{\succsim_d\}$ is Temporally Unbiased, $(x,\eta) \succsim^r (y, \eta\mu) \Leftrightarrow [x,t] \succsim_0 [y,s] \Leftrightarrow [x,t] \succsim_d [y,s] \Leftrightarrow (x, p(t|d)) \succsim^r (y, p(s|d)) \Leftrightarrow (x, \eta') \succsim^r (y, \eta'\mu)$. The proof for the other case in which $\eta' < \eta$ is similar.[7] ∎

The proof of Theorem 1 relies on the structural similarity between risky and intertemporal choices: a decrease in the risk is equivalent to the time of the reward and the decision time getting closer. Such a similarity had also been suggested by Prelec and Loewenstein (1991), although they did not provide a formal argument. The Probability Time Tradeoff axiom proposed by Baucells and Heukamp (2012) and used in Chakraborty (2016) to axiomatize preferences on a richer domain of intertemporal lotteries has a similar flavor.

[6] Since $(x, p(t|d)) = (x, p(t|d')p(d'|d)) \succsim^r (y, p(s|d')p(d'|d)) = (y, p(s|d))$. Since $(x, p(t|d)) \succsim^r (y, p(s|d))$, by the Independence Axiom, $(x, p(t|d')) \succsim^r (y, p(s|d'))$. Hence, $[x,t] \succsim_{d'} [y,s]$.

[7] Since p is strictly decreasing and bijection to $[0,1]$, there exist $t, s, d \in T$ such that $s \geq t$, $p(t) = \eta'$, $p(s) = \eta'\mu$, and $p(d) = \eta'/\eta$. Then, $p(t|d) = \eta$ and $p(s|d) = \eta\mu$. Since $\{\succsim_d\}$ is Temporally Unbiased, $(x,\eta) \succsim^r (y, \eta\mu) \Leftrightarrow (x, p(t|d)) \succsim^r (y, p(s|d)) \Leftrightarrow [x,t] \succsim_d [y,s] \Leftrightarrow [x,t] \succsim_0 [y,s] \Leftrightarrow (x, p(t)) \succsim^r (y, p(s)) \Leftrightarrow (x, \eta') \succsim^r (y, \eta'\mu)$.

B. *Results for Consumption Streams in Discrete Time*

In this subsection, we use equation (2) and the definitions in Table 1 to derive the characterization of discounting behavior through properties of the weighting function $g(\cdot)$.

REMARK 2: *Consider a DM represented by equation* (1) *with* $g(\cdot)$ *continuous on* $(0,1)$. *Let* $D(t)$ *be her composite discount function, as defined in equation* (2).

(i) *Diminishing Impatience holds if and only if for every* $r \in (0,1)$ *and* $t \in \mathbb{N}_+$:

$$g\big((1-r)^{t+1}\big) > g\big((1-r)\big)g\big((1-r)^t\big). \tag{10}$$

(ii) *Delay Independent Diminishing Impatience holds if and only if for every* $r \in (0,1)$ and $t,k \in \mathbb{N}_+$:

$$g\big((1-r)^{t+k}\big) > g\big((1-r)^k\big)g\big((1-r)^t\big). \tag{11}$$

(iii) *Strongly Diminishing Impatience holds if and only if for every* $r \in (0,1)$ *and* $t,t' \in \mathbb{N}$ *such that* $t < t'$:

$$\frac{g\big((1-r)^t\big)}{g\big((1-r)^{t+1}\big)} > \frac{g\Big((1-r)^{t'}\Big)}{g\Big((1-r)^{t'+1}\Big)}. \tag{12}$$

(iv) *Delay Independent Strongly Diminishing Impatience holds if and only if for every* $r \in (0,1)$, $t,t' \in \mathbb{N}, k \in \mathbb{N}_+$ *such that* $t < t'$:

$$\frac{g\big((1-r)^t\big)}{g\big((1-r)^{t+k}\big)} > \frac{g\Big((1-r)^{t'}\Big)}{g\Big((1-r)^{t'+k}\Big)}. \tag{13}$$

The proofs follow from the definitions. Next, we summarize the implications of risk attitude on intertemporal preferences in equation (1).

REMARK 3: *Consider a DM whose preferences are represented by equation* (1) *with* $g(\cdot)$ *continuous on* $(0,1)$.

(i) *Strict Certainty Effect* (6) *implies Delay Independent Diminishing Impatience* (*i.e.*, *equation* (11))

(ii) *Strict Common Ratio Effect* (5) *implies Strongly Diminishing Impatience* (*i.e.*, *equation* (12)) *and*, *hence*, *Delay Independent Strongly Diminishing Impatience* (*i.e.*, *equation* (13)).

The first claim holds by letting $p = (1-r)^k$ and $q = (1-r)^t$; the second claim holds by letting $p = 1-r$, $q = (1-r)^{t'-t}$, and $\ell = (1-r)^t$.

For the relation in the direction from time to risk, Diminishing Impatience as defined above does *not* imply Weak Certainty Effect for general weighting functions. The certainty effect implies a bias toward certainty *irrespective* of how risky the alternative is, the dual to which would be a bias toward the present $(t = 0)$ *irrespective* of the delay between the two prospects being compared. In evaluating the reason for the severed connection between time and risk preferences, we note that the definition of diminishing impatience used in the literature and defined in Table 1 focuses on a delay of a single period, thus only comparing $D(t)$ to $D(t+1)$ as t increases from 0. In our setting, this one-period definition is characterized by a particular property (equation (10)) of the weighting function $g(\cdot)$, that fails to generalize to the case of longer delays (equation (11)), even under technical assumptions of continuity and differentiability of $g(\cdot)$, as shown in our counterexample in online Appendix A. Thus diminishing impatience fails to account for present bias behaviorally. Theorem 2 below shows that Delay Independent Diminishing Impatience is sufficient for Weak Certainty Effect, and Strongly Diminishing Impatience is sufficient for Weak Common Ratio Effect.

THEOREM 2: *Consider a DM whose preferences are represented by equation* (1) *with continuous* $g(\cdot)$.

(*i*) *Strongly Diminishing Impatience implies Weak Common Ratio Effect.*

(*ii*) *Delay Independent Diminishing Impatience implies Weak Certainty Effect.*

PROOF:

(i) Assume Strongly Diminishing Impatience. By Remarks 1 and 2, it suffices to show that equation (12) in Remark 2 implies that equation (5) in Remark 1 holds with weak inequality and there exist p, q, ℓ for which equation (5) holds. Let $p = 1 - r$, $q = (1-r)^{t'-t}$, and $\ell = (1-r)^t$. Then equation (5) holds.

We will show that for any $p, q \in (0,1)$ and $\ell \in (0,1]$, equation (5) in Remark 1 holds with weak inequality. Since Strongly Diminishing Impatience and Delay Independent Strongly Diminishing Impatience are equivalent, in the following, we assume Delay Independent Strongly Diminishing Impatience. Consider a sequence $\{m_i/n_i\}_{i=1}^{\infty}$ of rational numbers that converges to $\ln p/\ln q\ell$, where m_i, n_i are positive integers. Similarly, consider a sequence $\{a_i/b_i\}_{i=1}^{\infty}$ of positive rational numbers that converges to $\ln\ell/\ln q\ell$, where a_i, b_i are positive integers. Note that $\ln\ell/\ln q\ell < 1$, so we can choose $\{a_i/b_i\}_{i=1}^{\infty}$ such that $a_i < b_i$. Given these sequences, define a sequence $\{r_i\}$, such that $q\ell = r_i^{n_i b_i}$, that is $r_i = (q\ell)^{1/(n_i b_i)} < 1$. Note that as a_i/b_i converges to $\ln\ell/\ln q\ell$, $r_i^{a_i n_i} = (q\ell)^{a_i/b_i}$ converges to $(q\ell)^{\ln\ell/\ln q\ell} = \ell$. Similarly, as m_i/n_i converges to $\ln p/\ln q\ell$, $r_i^{m_i b_i} = (q\ell)^{m_i/n_i}$ converges to $(q\ell)^{\ln p/\ln q\ell} = p$.

Using Delay Independent Strongly Diminishing Impatience, $\forall i$:

$$\frac{g\left(r_i^{a_i n_i}\right)}{g\left(r_i^{a_i n_i + m_i b_i}\right)} > \frac{g\left(r_i^{n_i b_i}\right)}{g\left(r_i^{n_i b_i + m_i b_i}\right)}.$$

Using the continuity of g, as $i \to \infty$, Weak Common Ratio Effect follows:

$$\frac{g(\ell)}{g(p\ell)} \geq \frac{g(q\ell)}{g(pq\ell)}.$$

(ii) Assume Delay Independent Diminishing Impatience. By Remarks 1 and 2, it suffices to show that equation (13) in Remark 2 implies that equation (6) in Remark 1 holds with weak inequality and there exist p, q for which equation (6) holds. Let $p = (1-r)^k$ and $q = (1-r)^t$. Then equation (6) holds.

Part (ii) is a special case of (i), where $\ell = 1$, $a_i = 0$, $b_i = 0$, and Delay Independent Diminishing Impatience replaces Delay Independent Strongly Diminishing Impatience. ∎

In the above theorem, we have Weak Common Ratio Effect and Weak Certainty Effect as behavioral implications, but not the versions with strict inequalities (Strict Common Ratio Effect and Strict Certainty Effect). This gap is inevitable given the difference between the connectedness in the domain of risk preferences (i.e, the probabilities are numbers in $[0,1]$) and the non-connectedness of the domain of time preferences (i.e., the dates are nonnegative integers).[8] The last step of the proof is to approximate a real number by the limit of rational numbers. When we take the limit, the related strict inequality inherited from behavior in the time domain becomes a weak inequality. We show in Corollary 1, that Strongly Diminishing Impatience implies Strict Common Ratio Effect for almost all probabilities; Diminishing Impatience implies Strict Certainty Effect for almost all probabilities.

COROLLARY 1: *Consider a DM whose preferences are represented by equation* (1) *with* $g(\cdot)$ *continuous on* $(0,1)$.

(i) *There exists a dense subset* Δ_1 *of* $(0,1)^2 \times (0,1]$ *such that Strongly Diminishing Impatience implies for any* $(p,q,\ell) \in \Delta_1$:

$$\frac{g(\ell)}{g(p\ell)} > \frac{g(q\ell)}{g(pq\ell)}.$$

(ii) *There exists a dense subset* Δ_2 *of* $(0,1)^2$ *such that Delay Independent Diminishing Impatience implies for any* $(p,q) \in \Delta_2$:

$$g(pq) > g(p)g(q).$$

PROOF:

For (i), define $\Delta_1 = \{(r^k, r^s, r^t) \mid r \in (0,1), k,s \in \mathbb{N}_+, t \in \mathbb{N}\}$. Notice that $\left(r_i^{m_i b_i}, r_i^{(b_i - a_i)n_i}, r_i^{a_i n_i}\right)$ in the proof of Theorem 2 is a sequence in Δ_1 that converges to $(p,q,\ell) \in (0,1)^2 \times (0,1]$. Part (ii) is proved similarly. ∎

[8] The proof of Theorem 1 in continuous time makes it clear that the complete relation, especially the relation from time preferences to risk preferences relies on the continuous time structure.

V. Discussion

In the temporal rewards setting, Assumption 1 implies that the passage of time affects the desirability of a prize only through the exponential accumulation of risk. Alternatively, one could have additionally allowed for "pure" time-discounting (earlier is better even when hazard rate $r = 0$). The risk-time correspondence in Theorem 1 would still hold in this alternative setting (details in online Appendix B) under certain separability conditions.

In the previous sections, we assume constant hazard rate, which implies preferences are Time-invariant (Halevy 2015). Under this assumption, we can deal with "static reversal" (violation of stationarity) and "dynamic reversal" (violation of time-consistency) interchangeably. If we allow for an arbitrary hazard rate (i.e., $r_t \neq r_s$ for calendar times $t \neq s$), static and dynamic reversals would no longer coincide. In online Appendix C we discuss how the results could be extended in that setting.

Moreover, neither assumption of "constant" or "arbitrary" hazard rate is more general in the context of our two-way results. Assuming the former provides a more general result when behavior over time implies behavior under risk (the direction which is missing in Halevy 2008, Saito 2011), and assuming the latter provides a more general result when behavior under risk implies behavior over time (the direction already established in the literature).

IV. Conclusion

This paper establishes a tight relation between nonstandard behaviors in the domains of risk and time, by considering a decision-maker with non-expected utility preferences who believes that only present consumption is certain while any future consumption is uncertain. In the domain of temporal rewards ($X \times \mathbb{R}_+$), we provide a complete relationship between risk and time preferences using two intuitive notions of time-behavior, Temporal Reversals and Present-Biased Temporal Reversals, which can be intrinsically linked to hyperbolic and quasi-hyperbolic discounting, respectively. For the choice domain of consumption streams ($X^{\mathbb{N}}$), our main result is that the notion of Diminishing Impatience does not imply Weak Certainty Effect (and hence also does not imply the strict version of Certainty Effect), *unless* it is adequately extended to hold for all possible delays between streams under consideration. Additionally, a stronger condition, Strong Diminishing Impatience implies Weak Common Ratio Effect, and also implies Strict Common Ratio Effect for almost all probabilities. In Table 2 we summarize how the current paper links to Halevy (2008) and Saito (2011).

Further, the temporal behaviors considered in the two domains (e.g., Present-Biased Temporal Reversals from $X \times \mathbb{R}_+$ and Delay Independent Diminishing Impatience from $X^{\mathbb{N}}$), are also interlinked, as shown in Proposition 3. Table 3 summarizes our results.

TABLE 2—CONNECTING THE PAPER'S RESULTS ON CONSUMPTION STREAMS TO PREVIOUS WORK

	Results linking risk and time in the Halevy (2008) setup						
Halevy (2008)	DI	Step 1 $\Leftrightarrow$	Functional inequality[a]	Step 2: Segal (1987) $\Leftrightarrow$ Lemma 4.1	Increasing elasticity of $g(\cdot)$	Step 3 $\Leftrightarrow$	SCE
Saito (2011)	(i) Unsubstantiates Segal's (1987) Lemma 4.1 used in Step 2 (ii) Proposes:						
	DI	Step 1 $\Leftrightarrow$ as before	Functional inequality of g[a]	Without $\Leftrightarrow$ using Steps 2–3	SCE		
This paper	Shows DI $\not\Rightarrow$ CE & DI $\not\Rightarrow$ Functional inequality[a] (online Appendix A). Shows Halevy (2008) and Saito (2011) results are uni-directional SCE $\Rightarrow$ DI. Establishes new two-way results (Theorems 1–2) summarized in Table 3. Introduces DIDI, a strengthening of DI. DIDI $\Rightarrow$ WCE and DIDI $\Rightarrow$ SCE for almost all probabilities.						

Note: Abbreviations used: Strict Certainty Effect (SCE), Weak Certainty Effect (WCE), Diminishing Impatience (DI), Delay Independent Diminishing Impatience (DIDI).

[a] A special case of Kahneman and Tversky (1979) subproportionality.

TABLE 3—SUMMARY OF THE PAPER'S RESULTS

Temporal rewards ($X \times \mathbb{R}_+$)			Interlink between temporal notions across domains			Consumption streams ($X^{\mathbb{N}}$)		
SCRE	Theorem 1 $\Leftrightarrow$	TR	TR	Proposition 3 $\Leftrightarrow$	SDI	SDI	Remark 3 $\Leftarrow$	SCRE
							Theorem 2 $\Rightarrow$	WCRE
SCE	Theorem 1 $\Leftrightarrow$	PBTR	PBTR	Proposition 3 $\Leftrightarrow$	DIDI	DIDI	Remark 3 $\Leftarrow$	SCE
							Theorem 2 $\Rightarrow$	WCE

Note: Abbreviations used: Strict Common-Ratio Effect (SCRE), Weak Common-Ratio Effect (WCRE), Temporal Reversals (TR), Present-Biased Temporal Reversal (PBTR), Strong Diminishing Impatience (SDI), Delay Independent Diminishing Impatience (DIDI).

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