

## Strotz Meets Allais: Diminishing Impatience and the Certainty Effect

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Consider a decision maker who prefers an immediate-smaller reward to a later-larger reward, but prefers the latter to the former when both alternatives are equally delayed. This decision maker is said to exhibit *diminishing impatience* (or *present bias*). It is a robust characteristic of many people's preferences, based both on introspection (Robert H. Strotz 1955) and numerous experimental studies (e.g., Richard Thaler 1981). A decision maker who makes future plans but does not take into account that, as time passes, the distant future becomes immediate, becomes more impatient, and will revise her choices as time progresses. Following this line of reasoning, diminishing impatience has become the leading experimental evidence brought to support time-inconsistent preferences. Diminishing impatience has been instrumental in the adoption of hyperbolic discounting, and particularly quasi-hyperbolic discounting (David Laibson 1997), which captures this regularity most succinctly. These formulations model diminishing impatience by assuming that the pure time preference, as measured by the discount rate, is highest at the present.

This paper presents an alternative perspective on diminishing impatience. It suggests that the crucial distinction between the present and the future is that only the present can be certain, while any future plan is uncertain. To model this asymmetry between the present and the future, the current study models the future as a random process that has a positive probability of stopping at any given period. This modeling strategy fits, among others, one uncertainty that can never be assumed away: uncertain lifetime. In this case, the termination probability may be interpreted as the hazard of mortality. Alternatively, it allows one to capture the risk that a promise of a future reward may be breached. In both cases, any path of planned consumption becomes random. If a decision maker evaluates this stochastic path using a utility function that is not linear in the continuation probability (that is, of the non-expected utility form<sup>1</sup>), she will exhibit diminishing impatience. The intuition behind this result is that under non-expected utility, preferences are disproportionately sensitive to certainty (Maurice Allais 1953; Daniel Kahneman and Amos Tversky 1979), which is a distinctive feature of the present. Therefore, the agent will exhibit present bias.

It is well known that in the realm of expected utility models, stopping probability (mortality risk) has a similar effect on time discounting as the pure time preference (Menahem E.

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<sup>1</sup> Experimental evidence lends strong support to theories in which decision makers evaluate risky prospects using utility functions that differ from expected utility (see Mark J. Machina 1987, Colin F. Camerer 1995, and Chris Starmer 2000 for surveys).

Yaari 1965). But as long as the pure time preference is constant, the choices are time consistent (Halevy 2005). In particular, models with constant termination probability are very popular in motivating geometric discounting (e.g., Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green 1995, 734). The current work introduces this uncertainty into a model of non-expected utility which is well known (at least since Howard Raiffa 1968) to generate dynamically inconsistent choices. The outcome is a simple and tractable representation, which is able to deliver the robust behavior of diminishing impatience, together with preference reversals of a myopic agent. In this representation, diminishing impatience is a result of the decision maker's perception of the future as uncertain.

It will be shown that existing experimental evidence (Gideon Keren and Peter Roelofsma 1995; Bethany J. Weber and Gretchen B. Chapman 2005) supports this study's methodology (while being inconsistent with other formulations) that the crucial cognitive distinction between the present and the future can be traced to the certainty of the former and the inherent uncertainty associated with the latter. Therefore, when the immediate becomes risky, present bias weakens substantially. These findings are consistent with the *Implicit Risk Approach* that associates delayed consequences with a subjective implicit risk. Once an explicit objective risk to immediate (as well as delayed) consequences is introduced, diminishing impatience weakens and becomes insignificant. The economic implications of this observation are far-reaching: if current decisions involve risk, impatience may not diminish. However, preference reversals that result from the temporal resolution of uncertainty may still be important. The current study presents a simple way to account for them.

The functional representation is based on Soo Hong Chew and Larry G. Epstein's (1990) axiomatization, but allows for the possibility of termination (due to mortality or disappearance) with a constant probability.<sup>2</sup> The essence of the representation is that for each state (stopping time, "lifespan"), the present discounted utility is calculated. Hence, when choosing the optimal path, the agent chooses among lotteries over present discounted utility. If time inconsistencies result from the randomness of future consumption, the utility function that represents preferences over such distributions must be nonlinear in the probabilities of different stopping times. Yaari's Dual Theory (Yaari 1987), which is usually used to evaluate lotteries whose outcomes are wealth levels, is applied to a domain in which outcomes are present discounted utility in different states of the world.

This work makes the essential link between the inherent uncertainty of the future (captured, for example, by lifetime) and non-expected utility. This link is necessary to derive a simple utility function that represents preferences over consumption paths, and can account for the experimental evidence. Furthermore, a parametric condition for diminishing impatience under this representation is derived. An identical condition accounts for the well-known behavioral patterns in choice under risk (Allais's common ratio effect) and under uncertainty (Daniel Ellsberg's ambiguity aversion). Furthermore, it is shown that, in the constant stopping probability case studied below, and for a very special case of uncertainty attitude, the representation reduces to the familiar quasi-hyperbolic discounting model—although the source of diminishing impatience is very different.

The paper is organized as follows. Section I provides a simple motivating example that shows how uncertain future might cause diminishing impatience and preference reversals. Keren and Roelofsma's (1995) findings are presented in Section II. They show that when the present becomes risky, the present bias weakens dramatically. Section III suggests a utility function that can account for the experimental evidence in an environment where the future is risky. This

<sup>2</sup> Although the constant probability of stopping (hazard) assumption is maintained throughout this paper, it can be easily relaxed—resulting in a more general (but less tractable) representation.

section includes a characterization of the necessary and sufficient condition for diminishing impatience and a discussion of some special cases. Section IV derives the utility function which was suggested in Section III by adapting Chew and Epstein's (1990) axioms to an environment with stopping probability (like uncertain lifetime), and adopting Yaari's Dual Theory. Section V discusses related literature and Section VI concludes with some thoughts for further research.

### I. A Motivating Example

Consider the following thought experiment comprising two choices. The subject is asked to choose between:

*Problem 1.* \$100 now or \$110 in 4 weeks;

*Problem 2.* \$100 in 26 weeks or \$110 in 30 weeks.

A subject who prefers an immediate \$100 in Problem 1, but \$110 in Problem 2, is said to exhibit diminishing impatience.<sup>3</sup> Now, assume that the probability in which the reward might disappear (or the probability of mortality) is 1 percent per week and the agent is time neutral.<sup>4</sup> The stopping probability captures the intuitive idea that the farther away in time is the promise, the lower is the probability it will be kept. The two temporal choice problems translate into a choice between prospects:

*Problem 1'.* (\$100, 1) or (\$110, 0.96);

*Problem 2'.* (\$100, 0.77) or (\$110, 0.74),<sup>5</sup>

where prospect  $(\$x, p)$  represents a lottery that pays  $\$x$  with probability  $p$  and  $\$0$  with probability  $1 - p$ . It has been well documented in experiments (Kahneman and Tversky 1979) that although expected utility theory predicts making the same choice (\$100 or \$110) in both, many subjects exhibit the *certainty effect*: they overweigh certain outcomes relative to very likely but not completely certain outcomes. As a result, they prefer \$100 in Problem 1' and \$110 in Problem 2'. This translates to \$100 today (at  $t = 0$ ) and \$110 in 30 weeks. When the certainty effect is present and the decision maker is myopic,<sup>6</sup> these choices would *not* be time consistent. Although today (at  $t = 0$ ), a subject prefers \$110 in 30 weeks, her preferences in 26 weeks (conditional on not stopping earlier, i.e., surviving) are to prefer \$100 immediately (at  $t = 26$  weeks) over \$110 in 4 weeks (at  $t = 30$  weeks).<sup>7</sup> These choices are often referred to as *preference reversals*.<sup>8</sup> The source of the certainty effect might be Ariel Rubinstein's (1988) similarity between the probabilities of realizing the prize in 26 and 30 weeks, while receiving the prize with certainty today is not

<sup>3</sup> Although the underlying representation derived in this work is in terms of consumption flows, the current example (as well as the following experiment) discuss monetary rewards. This is a general problem with the literature on diminishing impatience. A recent paper (Jawwad Noor 2007) studies the restrictions imposed on preferences when the expected consumption path is allowed to vary.

<sup>4</sup> These assumptions are made for expositional purposes only. If there is time discounting (in addition to uncertainty over the realization of the payment), the hazard rate could be lowered drastically.

<sup>5</sup> These prospects are derived from:  $(\$100, 0.99^{26})$  and  $(\$110, 0.99^{30})$ .

<sup>6</sup> Myopic implies that, in making current decisions, the agent does not take into account her future preferences and the decisions she will make.

<sup>7</sup> Since the hazard rate is constant at 1 percent per week, there is no place for learning, and the choice at  $t=26$  weeks between  $(\$100, \text{today})$  and  $(\$110, \text{in 4 weeks})$  translates into a choice between  $(\$100, 1)$  and  $(\$110, 0.96)$ . If the certainty effect is present, the subject prefers \$100 at  $t = 26$ , contrary to her preferences at  $t = 0$ .

<sup>8</sup> The analysis assumes preferences are consequential. That is, the decision in 26 weeks ignores the uncertainty borne by the subject up to that point in time. Machina (1989) suggests relaxing consequentialism in order to maintain dynamic consistency. See also Peter Wakker (1997).

perceived as similar to the probability of receiving it in 4 weeks. Although the certainty effect does not necessarily create monotonically diminishing impatience, it does create a pointwise similar effect when comparing the present to the future: that is, the willingness to sacrifice later consumption for earlier consumption is highest at the present. Hence, it fits the experimental evidence that impatience is diminishing, especially for short horizons (Uri Benzion, Amnon Rapoport, and Joseph Yagil 1989; Daniel Read 2001; Shane Frederick, George Loewenstein, and Ted O'Donoghue 2002).

## II. Experimental Evidence

Before proceeding to a functional representation, it is worthwhile to test whether the explanation advocated here fits behavior in experiments better than standard models that account for diminishing impatience. For simplicity, denote a two-outcome temporal lottery by  $((x, t), p)$ , where  $p$  denotes the probability of payment  $x$  in period  $t$ , and with probability  $1 - p$  the subject is paid nothing.<sup>9</sup> Keren and Roelofsma (1995, experiment 1) divided their subjects into six groups. The first two groups faced the standard "preference reversal" problem, as in Problems 1 and 2. The first group chose between  $((100, 0), 1)$  and  $((110, 4), 1)$ , where  $x$  is measured in guilders (Fl.) and  $t$  is measured in weeks. The second group chose between  $((100, 26), 1)$  and  $((110, 30), 1)$ . The difference between the two groups is that both payments in the second group are delayed by 26 weeks. The next four groups were offered similar temporal alternatives, but probabilistic. For groups 3 and 4, the probability of realizing the payment was lowered to 90 percent. That is, group 3 had to choose between  $((100, 0), 0.9)$  and  $((110, 4), 0.9)$  and group 4 had to choose between  $((100, 26), 0.9)$  and  $((110, 30), 0.9)$ . For groups 5 and 6, the probability of the temporal payments was further reduced to 50 percent. Table 1, reproduced from Keren and Roelofsma (1995), summarizes their findings.

As the table clearly shows, when certainty is reduced, the proportion of subjects choosing the immediate but smaller reward decreases significantly from 82 percent when rewards are certain to 39 percent when the probability of rewards is 0.5. As a result, when the present becomes risky, present bias weakens drastically. Weber and Chapman<sup>10</sup> (2005) replicated Keren and Roelofsma's (1995) findings.

None of the existing theories of intertemporal choice and, in particular, those that account for diminishing impatience through an accommodation of the discount factor, can account for this experimental evidence. The reason is that if preferences are monotone (as in any reasonable theory of choice under risk used in conjunction with those intertemporal theories) and the subject prefers  $((100, 0), 1)$  to  $((110, 4), 1)$ , then she must prefer  $((100, 0), 0.5)$  to  $((110, 4), 0.5)$ .<sup>11</sup> As is evident from the experiment, this is not the case.

TABLE 1—IMPATIENCE OVER RISKY OUTCOMES

	Probability of monetary reward		
	$p = 1.0$	$p = 0.9$	$p = 0.5$
<i>Immediate future</i>			
A. Fl. 100 now	0.82	0.54	0.39
B. Fl. 110 in 4 weeks	0.18	0.46	0.61
<i>Remote future</i>			
C. Fl. 100 in 26 weeks	0.37	0.25	0.33
D. Fl. 110 in 30 weeks	0.63	0.75	0.67
<i>Difference</i>	0.45	0.29	0.06

Source: Keren and Roelofsma 1995.

<sup>9</sup> Note that, in principle, it could matter when the uncertainty is resolved: the subject might not be indifferent between learning today the outcome of the lottery or learning it only in period  $t$ . The current paper does not attempt to integrate this preference into the subject's preferences; however, it is an important extension.

<sup>10</sup> The reader is referred to Experiment 2 (summarized in Tables 5 and 6) in their study.

<sup>11</sup> In other words, assuming that pure nonconstant time preference accounts for column 1 ( $p = 1$ , choice under certainty) and non-expected utility accounts for weakening the present bias (columns 2 and 3) is inconsistent with any reasonable theory of choice under uncertainty.

The results of Keren and Roelofsma (1995) and Weber and Chapman (2005) are consistent with the explanation of diminishing impatience presented in the current study,<sup>12</sup> based on the subject's belief of the implicit uncertain future (taken as geometric, with stopping probability of  $r$ ), the reward of \$110 in 4 weeks *cannot* be certain. The subject transforms  $((110, 4), 1)$  to the prospect  $(110, (1 - r)^4)$ . That is, when comparing \$100 today to \$110 in 4 weeks, the subject compares a certain \$100 to a risky \$110. However, when comparing a lottery that pays \$100 today with probability 0.5, to a lottery that pays \$110 in 4 weeks with probability of 0.5, the subject compounds the two risks she faces and compares:  $(\$100, 0.5)$  and  $(\$110, 0.5 \cdot (1 - r)^4)$ . In this case the present is risky and, as a result, present bias disappears. This experiment demonstrates that the crucial property of the present, which is responsible for diminishing impatience, is its certainty.

### III. Diminishing Impatience

This section generalizes the motivating example presented in Section I, which showed the close relationship between evaluation of temporal payoffs in an environment with uncertain lifetime and non-expected utility preferences (and especially the certainty effect). It proposes a simple and tractable utility function (which will be derived in the next section) that can account for the experimental evidence of Keren and Roelofsma (1995) and Weber and Chapman (2005). The discussion formally characterizes diminishing impatience and shows that the same parametric condition that accounts for the common ratio effect (Allais 1953) and ambiguity aversion (Ellsberg 1961) can account for diminishing impatience as well. The section concludes with a discussion of special cases of the utility function proposed.

Denote by  $c_t$  period  $t$ 's consumption, and by  $\mathbf{c} = (c_0, c_1, c_2, \dots)$  a consumption path. Assume the only risk the agent faces is the risk that the consumption flow may be truncated (which captures both mortality and the risk of disappearance). Denote by  $r$  the constant stopping probability (e.g., death or disappearance). This probability captures the intuitive idea of *implicit risk value*: the farther away in time is a promise, the lower is the probability it will be kept. The agent's utility of consumption path  $\mathbf{c}$ , which is subject to a constant stopping probability (hazard) of  $r$ , is:

$$(1) \quad U_Y(\mathbf{c}, r) = \sum_{t=0}^{\infty} g((1 - r)^t) \beta^t u(c_t),$$

where  $u(t)$  is the agent's felicity function,<sup>13</sup>  $\beta$  is the constant "pure" time preference, and  $g$  is an increasing function from the unit interval to itself, satisfying  $g(0) = 0$ ,  $g(1) = 1$ . Note that if  $r = 0$  (a deterministic consumption path), the intertemporal utility function exhibits geometric (constant) discounting. Therefore, uncertainty is not the unique source of time discounting, and this paper should not be viewed as attempting to derive time preference from uncertainty. For  $r > 0$ , if  $g(\cdot)$  is the identity function,  $U_Y(\mathbf{c}, r)$  reduces to the standard separable expected-utility intertemporal utility function (Yaari 1965; Halevy 2005). In this case, time discounting of period  $t$  is the product of pure time preference— $\beta^t$ , and the probability the consumption path will not be truncated before  $t$ , which equals  $(1 - r)^t$ . The representation (1) permits the transformation of  $(1 - r)^t$ , which is the probability the consumption will not be stopped before period  $t$ , using  $g(\cdot)$ . Convexity of  $g(\cdot)$  implies that while the weight of current (and certain) consumption in the utility

<sup>12</sup> Again, as in the previous section, time neutrality is assumed for expositional purposes. As will be clear from the following section, this assumption is nonessential.

<sup>13</sup> The axiomatic derivation requires that  $u(\cdot)$  exhibit constant relative risk aversion (CRRA), with a coefficient of relative risk aversion smaller than one.

function is not altered, the risk that any future consumption will not materialize is amplified, since  $g(1) = 1$  and  $g((1 - r)^t) < (1 - r)^t$ . Therefore, convexity of  $g(\cdot)$  is necessary in order to capture the certainty effect in the motivating example.

Let  $D(t)$  denote the decision maker's *time discounting* of period  $t$ , which is composed of her pure time preference (represented by her discount rate— $\beta$ ) and the transformation of the probability the consumption flow will not stop before period  $t$ . Then:

$$(2) \quad D(t) = \beta^t g((1 - r)^t).$$

As can easily be seen, future uncertainty is not the unique source of time discounting. In principle, one could assume that the pure time preference is not constant as well (as in the quasi-hyperbolic case). The experimental evidence shown in Table 1 tends not to support this generalization. If nonconstant pure time preference was a major source of diminishing impatience, present bias would not weaken so drastically once explicit risk (which affects only the certainty effect) has been introduced.

The decision maker's *impatience at period  $t$*  is:

$$(3) \quad I(t) := \frac{D(t)}{D(t + 1)}.$$

$I(t)$  represents the agent's willingness to substitute utility at  $t + 1$  for utility at  $t$ .

**DEFINITION 1:** *The decision maker exhibits diminishing impatience if:  $I(0) > I(t)$  for every  $t \geq 1$ .*

Note that diminishing impatience as defined above does not imply that  $I(\cdot)$  is a monotonically decreasing function of  $t$ . One can define a decision maker to have *strongly diminishing impatience* if:  $I(0) > I(t) > I(t + 1)$  for every  $t \geq 1$ . As reported in Frederick, Loewenstein, and O'Donoghue (2002), the empirical evidence for strongly diminishing impatience is weak. Furthermore, Meribet Coller, Glenn W. Harrison, and E. Elisabet Rutström (2003) find strong experimental support of constant time discounting when the shorter horizon payment occurs in the future. It is easy to verify that standard geometric discounting implies constant impatience; quasi-hyperbolic discounting (Laibson 1997) implies diminishing impatience but not strongly diminishing impatience, and hyperbolic discounting implies strongly diminishing impatience. The following theorem characterizes the condition for diminishing impatience (which is supported empirically) for preferences represented by (1).

**THEOREM 1:** *Let the decision maker's preferences be represented by the utility function  $U_Y(\mathbf{c}, r) = \sum_{t=0}^{\infty} g((1 - r)^t) \beta^t u(c_t)$ . She exhibits diminishing impatience if and only if the elasticity of  $g(\cdot)$  is increasing.*

**PROOF:**

From (1):  $I(0) = 1/[\beta g(1 - r)] > \beta^t g((1 - r)^t)/[\beta^{t+1} g((1 - r)^{t+1})] = I(t)$  if and only if  $g((1 - r)^{t+1}) > g(1 - r)g((1 - r)^t)$  since this should hold for all  $t$  and  $r$  it holds if and only if  $g(\cdot)$  satisfies that for all  $0 < p, q < 1$ :  $g(p \cdot q) > g(p) \cdot g(q)$ . Uzi Segal (1987a, Lemma 4.1) showed this holds if and only if the elasticity of  $g$ , defined by  $\varepsilon_{g,p} := pg'(p)/g(p)$ , is increasing.

Increasing elasticity of the transformation function  $g(\cdot)$  turns out to be a critical feature in accounting for at least three empirical regularities of decision makers. The common ratio effect

(Allais 1953) in choice under risk could be accounted for by Rank Dependent Utility (of which the Dual Theory, which is used here, is a special case) if and only if the elasticity of  $g(\cdot)$  is increasing (Segal 1987b, Theorem 1). Segal (1987a, Theorem 4.2) showed that a resolution of the Ellsberg paradox that is based on relaxing the reduction of the compound lotteries assumption (which descriptively is probably the case; see Halevy 2007) requires a convex  $g(\cdot)$  to have increasing elasticity. The current study finds that an increasing elasticity of  $g(\cdot)$  accounts for diminishing impatience as well.

#### A. Special Cases

It may be of interest to point out that other utility functions which have been used in the literature can be obtained as a special case of (1). For example, suppose that  $g(\cdot)$  is a simple capacity (James Dow and Sergio Riberio da Costa Werlang 1992): a contraction of an additive probability measure by a constant parameter  $\gamma$ , except that the whole sample space preserves the measure of 1. Formally, given an additive probability measure  $\pi$ :

$$(4) \quad g(\pi) = \begin{cases} \gamma\pi & \pi < 1 \\ 1 & \pi = 1 \end{cases} \quad \text{for } \gamma < 1.$$

The simple capacity above could be motivated by constant uncertainty aversion (as in Dow and Werlang 1992). In this case, the decision maker does not know exactly what is the stopping probability and when she faces this ambiguity she behaves in a conservative fashion. Alternatively, it could capture the certainty effect found by Kahneman and Tversky (1979). After substitution into (1), the resulting utility function is

$$(5) \quad U_Y(c, r) = u(c_0) + \gamma \sum_{t=1}^{\infty} ((1-r)\beta)^t u(c_t),$$

which resembles the  $(\beta, \delta)$  preferences used by E. S. Phelps and R. A. Pollak (1968), Laibson (1997), Christopher Harris and Laibson (2000), O'Donoghue and Matthew Rabin (1999), and many others. It should be emphasized, however, that this is a very special case; not only in the uncertainty attitude of the decision maker (represented by (4)), but also as it assumes that the decision maker's belief may be summarized by a geometric model of stopping. Furthermore, the source of diminishing impatience in (5) is the risky future ( $r > 0$ ), while the pure time preference ( $\beta$ ) is constant.

Another function that has been suggested in the literature is

$$(6) \quad g(\pi) = \frac{e^\pi - 1}{e - 1}.$$

This function is able to accommodate the Allais and the Ellsberg paradoxes in decisions under risk and uncertainty (Segal 1987a). The resulting intertemporal utility function is

$$(7) \quad U_Y(c, r) = u(c_0) + \frac{1}{(1-e)} \sum_{t=1}^{\infty} \beta^t e^{(1-r)^t - 1} u(c_t).$$

It is easy to confirm that this function exhibits strongly diminishing impatience.

#### IV. Functional Representation

In this section, the representation (1) is derived from basic axioms on preferences over consumption paths, and an auxiliary assumption on the decision maker's belief over future uncertainty. It builds on a modification of Chew and Epstein's (1990) functional representation of a decision maker's preferences over stochastic consumption paths. It then derives (1) by adopting Yaari's (1987) Dual Theory (which satisfies the representation) and by considering consumption paths that are subject to a constant stopping probability.

The general framework characterizes decision makers' preferences that are defined over probability distributions on bounded and nonnegative consumption paths. Chew and Epstein's (1990) modified axioms and representation theorem are presented in the Appendix. The axioms imply that the decision maker's "pure" time preference ( $\beta$ ) is constant and that preferences are increasing with respect to a temporal generalization of stochastic dominance. Furthermore, time inconsistency is allowed *only* if the present or next period's consumption levels are uncertain.

For every deterministic consumption path  $c = (c_0, c_1, \dots, c_t, \dots)$ , let  $\sum_{t=0}^{\infty} \beta^t u(c_t)$  be the present discounted utility. For a probability distribution  $p$  over possible consumption plans, let  $\phi(p)$  denote the induced cumulative distribution function (CDF) of the present discounted utility. Theorem 0 (in the Appendix) shows that preferences over stochastic consumption paths satisfy the axioms above if and only if they are represented by a certain class of functions that are defined over the CDF of the present discounted utility.

To understand the representation result, consider a random future path represented by a specific probability distribution  $p$ , which induces a probability distribution (lottery) over present discounted utility with a CDF denoted by  $\phi(p)$ . This lottery over present discounted utility is evaluated using a functional  $V$ , and the invariance properties (see Appendix) restrict the set of admissible functionals. Although, in principle, it is possible to apply this general representation to economic environments (e.g., Chew and Epstein 1990), it is helpful to see what functionals  $V$  can satisfy the invariance properties. Furthermore, to relate this general representation to the empirical evidence on diminishing impatience, additional structure is required.

Two examples of admissible functionals  $V(\cdot)$  are the Expected Value functional and Yaari's (1987) Dual Theory. The Expected Value functional is given by:

$$(8) \quad U_E(p) = E_p \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{where } u(c_t) = \frac{c_t^{1-\alpha}}{1-\alpha} \quad \alpha < 1,$$

where  $U_E(p)$  denotes the utility from a probability distribution  $p$  over consumption paths. Although the expected value representation has been used extensively in the literature, its limitations are well known. In particular, relative risk aversion and the elasticity of intertemporal substitution are both functions of  $\alpha$ . Hence, comparative statics corresponding to changes in  $\alpha$  do not have clear interpretation (for a more detailed discussion see Epstein and Stanley E. Zin 1989). Furthermore, preferences represented by the expected value representation will be time consistent, that is, will satisfy Chew and Epstein's (1990) recursivity.

As an alternative to expected value, consider Yaari's (1987) Dual Theory. To facilitate the understanding of this theory, consider a simple lottery with monetary outcomes:  $A = (x_1, p_1; x_2, p_2; \dots; x_n, p_n)$  such that outcome  $x_i$  is received with probability  $p_i$  ( $i = 1, \dots, n$ ) and  $x_1 \leq x_2 \leq \dots \leq x_n$ . Denote by  $F_A$  the CDF of  $A$ . The utility of such a lottery is represented by the following functional:

$$(9) \quad V_Y(F_{(x_1, p_1; x_2, p_2; \dots; x_n, p_n)}) = x_1 + \sum_{i=2}^n (x_i - x_{i-1}) g\left(\sum_{j=1}^n p_j\right),$$



where  $g: [0, 1] \rightarrow [0, 1]$  such that  $g(0) = 0$ ,  $g(1) = 1$ , is increasing, and risk aversion implies convexity of  $g(\cdot)$ .<sup>14</sup> This representation is linear in outcomes (hence satisfies the invariance conditions in Theorem 0), but transforms the decumulative distribution  $(I - F_A)$  using  $g(\cdot)$ . This representation is a special case of Choquet expected value, and therefore can be viewed as a special case of David Schmeidler's (1989) and Itzhak Gilboa's (1987) theory of expected utility with respect to nonadditive measure. This is an important point: although Theorem 0 is stated in terms of objective probabilities, transformation functions that are different from the identity function can be interpreted as a consequence of the decision maker's aversion to ambiguity (see Yaari (1987) on the relation between his theory and Choquet expected utility).

To gain more intuition about this representation and the role of  $g(\cdot)$ , consider the lottery  $(x_1, p; x_2, 1 - p)$  such that  $x_1 \leq x_2$ . Then,

$$V_Y(F_{(x_1, p; x_2, 1-p)}) = x_1 + (x_2 - x_1)g(1 - p) = x_1(1 - g(1 - p)) + x_2g(1 - p).$$

If  $g(p) \leq p$ , then  $V_Y$  represents a pessimistic evaluation of the lottery, because  $x_1$  is evaluated using a higher weight than  $p$ , and  $x_2$  is evaluated using a lower weight than  $(1 - p)$ . The Yaari functional can be applied easily to represent intertemporal preferences by

$$(10) \quad U_Y(p) = V_Y(\phi(p)).$$

While this is a generalization of (8), it has the virtue that, contrary to (8), it is possible to separate risk aversion (captured by  $g(\cdot)$ ) from intertemporal substitution (which is a function of  $\alpha$ ). Furthermore, this generalization permits time inconsistent preferences.

In the presence of the axioms underlying Theorem 0, it is not hard to see that an axiomatic foundation of (10) could be derived, similarly to Yaari (1987) and Schmeidler (1989), by a weakening of the independence axiom to comonotonic independence.<sup>15</sup>

Note that although the axiomatic framework of Chew and Epstein (1990) permits all possible probability distributions over bounded consumption paths, the following section considers only geometric probability distributions, which reflect a constant stopping or disappearance probability. The more restrictive domain enables the compact representation (1) in which non-expected utility preferences manifest themselves in diminishing impatience.

#### A. Disappearance (or Mortality) Risk

We are now ready to derive the representation (1) suggested in Section III, by utilizing Yaari's functional (10) to represent the agent's preferences over consumption paths, when her belief over future paths is described by a sequence of stopping probabilities (which captures both the hazard of mortality and the risk of disappearance). This process captures the intuitive idea of *implicit risk value*: the farther away in time is a promise, the lower is the probability it will be kept.

Assume (for simplicity only) that the probability of stopping (the hazard rate) in period  $t$ , denoted by  $r(t)$ , is independent of time and equals  $r$ . Objects in this (restricted) choice set are elements  $(c, r)$  where  $c$  is a deterministic consumption plan and  $r$  is the (constant) stopping

<sup>14</sup> For obvious reasons Yaari's Dual Theory belongs to the "Rank Dependent Utility" family of non-expected utility functions.

<sup>15</sup> Two random variables are comonotonic if, for any two states of the world, neither of them can be a hedge against the other (they are "bets on the same event"). The condition of comonotonic independence states that if  $p$ ,  $p'$ , and  $p''$  are pairwise comonotonic, then for every  $0 \leq \alpha \leq 1$ :  $p \succeq p'$  implies that:  $\alpha p + (1 - \alpha)p'' \succeq \alpha p' + (1 - \alpha)p''$ .

probability. The lottery (in utility terms) the agent faces in this case from consumption path  $\mathbf{c} = (c_0, c_1, \dots, c_t, \dots)$  is:

$$(11) \quad \left( u(c_0), r; u(c_0) + \beta u(c_1), r(1-r); \dots; \sum_{i=0}^t \beta^i u(c_i), r(1-r)^{t-1}; \dots \right).$$

Note that the payoffs here are naturally ordered: the longer the stopping time (e.g., lifetime), the higher is the agent's present discounted utility from any given consumption path.<sup>16</sup> The following theorem gives a representation of the agent's preferences over consumption paths when  $V(\cdot)$  is Yaari's dual theory utility function and the decision-maker belief over the consumption process can be summarized by a constant probability of stopping.

**THEOREM 2:** *Let all the assumptions of Theorem 0 hold, let  $V(\cdot)$  be Yaari's Dual theory functional (or Choquet expected value), and assume that the agent's belief over the consumption process can be characterized by a constant stopping probability which equals  $r$ . Then, the agent's preferences over consumption flows are represented by the following function:*

$$(12) \quad U_Y(\mathbf{c}, r) = \sum_{i=0}^{\infty} g((1-r)^i) \beta^i u(c_i) \quad \text{and} \quad u(c_i) = \frac{c_i^{1-\alpha}}{1-\alpha} \quad 0 < \alpha < 1.$$

**PROOF:**

Apply Theorem 0, the Dual Theory (9), and the specific stochastic process induced by uncertain lifetime (11):

$$\begin{aligned} U_Y(\mathbf{c}, r) &= V_Y(\phi(\mathbf{c}, r)) = u(c_0) + [(u(c_0) + \beta u(c_1)) - u(c_0)]g(1-r) + \dots \\ &= \sum_{i=0}^{\infty} g((1-r)^i) \beta^i u(c_i). \end{aligned}$$

As noted above, when  $g(\cdot)$  is the identity function, then (12) reduces to the standard discounted utility model (8) and the continuation probability enters symmetrically with time preference into the intertemporal utility function. Identification of the cause for diminishing impatience rests in this case on separation of the two ingredients of time discounting (Halevy (2005) focuses on this point), but does not allow the discussion of time inconsistency issues.

Theorem 2's representation can be easily generalized to nonstationary stopping (i.e., nongeometric distribution of stopping). However, diminishing impatience in that case will be distribution-dependent, and will depend not only on  $g(\cdot)$ . For more general stochastic processes (not necessarily of stopping), one has to resort to (10) which is time inconsistent. However, the connection to diminishing impatience is, then, not immediate.

## V. Related Literature

Several authors (Drazen Prelec and Loewenstein 1991; Loewenstein and Prelec 1992; Read 2001; Rubinstein 2003; John Quiggin and John Horowitz 1995) have noticed that the descriptive problems the constant time preference model faces resemble those faced by the Expected Utility model. These authors borrowed tools developed for the uncertain environment to analyze the

<sup>16</sup> Hence, all lotteries over consumption paths are comonotonic.

*separate problem* of temporal decision making under certainty. Howard Rachlin (Rachlin et al. 1986; Rachlin and Andres Raineri 1992; Rachlin, Jay Brown, and David Cross 2000) notes that these problems have similar structure and shows that if probability is interpreted as waiting time, the hyperbolic discount function can account for some of the observed behavior in choice under risk (e.g., the certainty effect). The current work, however, views the decision maker's perception and choice in risky environments as the underlying cognitive process that shapes her diminishing impatience. Hence, it draws a causal relation between the two. This study brings experimental evidence (Keren and Roelofsma 1995; Weber and Chapman 2005) to support this view.

The theory and experimental findings presented in this work are consistent with the *Implicit Risk Approach* (Julian B. Rotter 1954; Alvin R. Mahrer 1956; W. Mischel and J. Grusec 1967). This hypothesis is part of Rotter's Social Learning Theory (Rotter 1954). Rotter claimed that people choose the course of action (behavior) that maximizes a function of utility of possible outcomes and the *subjective* probability that these outcomes will be attained.<sup>17</sup> This theory emphasizes that an individual's choices and preferences (personality) represent the interaction of the person with her environment. Life experience constructs a certain set of subjective beliefs used in the evaluation of alternative actions. When life experience changes, the evaluation of behavior can change. As the agent accumulates life experience, the harder (but not impossible) it becomes to adjust it. In particular, the subjective probability (which is based on experience) can be different from the objective probability a consequence will occur. In the context of immediate versus delayed rewards, a person may assign a subjective probability (based on experience) that a promise of a delayed reward will not be kept. Support for this interpretation can be found in Mahrer (1956), who showed that strengthening children's trust in the promise maker increases the frequency of their choosing delayed rewards over immediate rewards. The theory presented in this paper is consistent with Rotter's framework: present bias may exist, but it is wholly due to the fact that the present is certain and the future is uncertain.

Takashi Hayashi (2003) provides axiomatic foundations for quasi-hyperbolic preferences by utilizing Epstein's (1983) framework of preferences over stochastic consumption paths. He relaxes stationarity and allows preferences over paths to change as a result of delaying them from the present to the future. To understand the relation to the current work, it is important to point out that Hayashi does not seek an underlying factor that hampers stationarity and assumes the decision maker evaluates the random paths using expected utility.

It is of interest to compare this paper's methodology to the recent work of Efe A. Ok and Yusufcan Masatlioglu (2007). They provide a representation for preferences on the prize-time space (like Peter C. Fishburn and Rubinstein 1982) under full certainty that relies on the weakening of transitivity while maintaining separability between disutility of time delay and utility of outcomes. Their representation encompasses, inter alia, exponential discounting (when stationarity and transitivity are imposed), hyperbolic discounting, Read's (2001) subadditive discounting, and Rubinstein's (2003) procedural similarity. Note that in their deterministic framework, diminishing impatience could be accounted for only by relaxing stationarity or transitivity, while the current work assumes geometric (exponential) discounting under full certainty. The diminishing impatience here relies on the uncertainty of delayed consequences and their evaluation using a functional that is nonlinear in probabilities. Furthermore, the domain of preferences here is random consumption paths. Ok and Masatlioglu (2007) and the current framework are complementary and present different aspects of the same phenomenon. In principle, one could write a meta-model that would encompass both; but this is not an easy task since the domains on which preferences are defined in the two models are different. Here it is consumption paths, and

<sup>17</sup> Note that Rotter did not claim that this function is linear in the subjective probabilities.

in Ok and Masatlioglu (2007) it is the prize-time space. Furthermore, the experimental evidence presented above tends not to support this generalization.

Another work that relies on the asymmetry between the present and the future is that of Jesús Fernández-Villaverde and Arijit Mukherji (2002). They present a model where preferences are shocked in every period but the agent learns about the current shock before consumption decisions are made. They show that, in this framework, different agents (who receive different shocks during the current period) may make different decisions when deciding between immediate and delayed consumption, but as the time horizon is shifted to the future, the current different shocks become irrelevant and all agents will make the same choice. A critical difference between the two models is that in the Fernández-Villaverde and Mukherji's (2002) framework the decisions are time consistent (and they present experimental evidence that shows that the demand for commitment devices is quite limited), while, here, myopic decisions are time inconsistent.

As noted above, for an agent with expected utility preferences, mortality risk has been used extensively to motivate exponential discounting. Halevy (2005) has shown that time consistency (which requires exponential discounting) does not impose any restrictions on the hazard rate. This observation allows one to relate decreasing hazard to diminishing impatience but *cannot* create preference reversals. Several authors (Peter D. Souza 1998; Omar Azfar 1999) presented models that can generate a decreasing hazard rate due to Bayesian updating of an unknown constant hazard (which is limited in its empirical validity); and Halevy (2005) has shown that a similar effect can be achieved with an unknown increasing hazard rate as well. Antoine Bommier (2004) introduces aversion to early death as a component of the agent's subjective discount rate. He shows that, if the decision maker is risk averse with respect to her length of life, this component will be decreasing in time and may lead the agent to exhibit time-inconsistent behavior. Partha Dasgupta and Eric Maskin (2005) try to rationalize hyperbolic discounting in an environment in which payoffs may be realized early. They show that the decision maker becomes more impatient as the horizon is shortened, since the likelihood of early realization diminishes in time. Gary S. Becker and Casey B. Mulligan (1997) present a framework in which the agent can invest costly resources that will decrease her time preference, hence, endogenizing the subjective discount rate. In their framework, a decrease in expected mortality will increase patience, not only through the direct effect of a decrease in the hazard rate, but also through generating an incentive to become more patient, since the expected return on this investment increases.

As noted in the introduction, it is well known that non-expected utility preferences might produce dynamically inconsistent choices. This observation has motivated a substantial research program that studies the consequences of this inconsistency (e.g., Edi Karni and Zvi Safra 1989; Green 1987), how to model dynamic choice when the independence axiom is relaxed (Machina 1989; Karni and Schmeidler 2001; Epstein and Michel Le Breton 1992; Peter Wakker 1997), and questions whether dynamic consistency is a desirable normative criterion in modeling choice under risk (Yaari 1985).

## VI. Conclusion

Diminishing impatience is a robust experimental finding. This paper suggests (and provides experimental support) that the critical component in accounting for it is the certainty of the present versus the uncertainty associated with a delayed consequence. Only preferences that are representable by a non-expected utility function are appropriate to apply in modeling diminishing impatience that results from this distinction between the present and the future, since they are disproportionately sensitive to certainty. The functional representation derived (1) is simple and easy to apply to a variety of economic environments (e.g., Halevy 2004). Furthermore, this

approach enables one to link diminishing impatience to other behavioral regularities in the field of choice under risk and uncertainty, as the common ratio effect and the Ellsberg paradox.

A natural extension of the current model could shed some light on the problem of procrastination. The literature on procrastination has long emphasized that when it comes to payments (or negative payoffs) most decision makers would prefer to substitute an immediate, small cost for a later, larger cost, but will refuse to do this substitution when an equal delay is introduced to the two alternatives. Consider a worker who is required to meet a deadline in completing a task. When there is plenty of time, she prefers to finish it before the deadline to minimize the stress associated with “last minute” completion. As time passes, however, and the deadline approaches, she tends to procrastinate and delays the completion of the task.<sup>18</sup> If one translates this problem to the current framework using a constant probability that the cost might disappear,<sup>19</sup> it translates to risk-seeking behavior when the horizon is short. This is consistent with Kahneman and Tversky’s (1979) findings of the *Reflection Effect*: people become risk loving in the neighborhood of certainty when negative payoffs (relative to the status quo) are involved. Application of Cumulative Prospect Theory (Tversky and Kahneman 1992), which is a generalization of rank dependent utility that allows for the reflection effect, could account for procrastination in taking necessary actions with negative consequences.

An important extension of the current work should include a rigorous treatment of subjective uncertainty (alluded to by the implicit risk approach). The experimental evidence shown in Section II used objective probability, and for this reason the axiomatic derivation of (1) is stated in terms of objective probabilities. Although, as mentioned in the text, the representation may be interpreted in terms of ambiguity aversion, the compounding of objective and subjective uncertainties in that case is not obvious (see Halevy 2007). I believe that a more satisfying avenue would be to include a subjective state space (as in Eddie Dekel, Barton L. Lipman, and Aldo Rustichini 2001) that will capture the implicit risk. The challenging problem would be to derive a representation that is not linear in probabilities (hence subject to the certainty effect), but able to handle objective states (risk) in a consistent way. This is an exciting avenue of research that is definitely worth pursuing.

As noted in the introduction, the benchmark of the current paper is geometric discounting under certainty. Although adopting this benchmark focuses attention on time inconsistencies that result from uncertain future (stationarity), a more flexible model could take as a baseline more general preferences (like Ok and Masatlioglu 2007), and similarly study the effect of introducing uncertainty there. In this context, it is important to emphasize that the reasoning presented in the current paper is not intended to capture all cases where self-control problems may arise. For example, Faruk Gul and Wolfgang Pesendorfer (2001, 2004) study a general framework where tastes may change over time. One source of a self-control problem may be diminishing impatience, which is the focus of the current study, but other sources (e.g., temptation) are possible, too, and the structure of the decision problem in those cases may be different from the one studied here.

#### APPENDIX: THE AXIOMATIC STRUCTURE

This appendix presents a modification to Chew and Epstein’s (1990) functional representation of a decision maker’s preferences over stochastic consumption paths. It is worth noting the two main differences from Chew and Epstein (1990): present consumption is allowed to be random (while in Chew and Epstein the present is always certain), and the choice set includes paths with

<sup>18</sup> An excellent example is given by O’Donoghue and Rabin (1999) using the chosen time to file a tax return.

<sup>19</sup> For example, if a coworker might complete the task and she will not have to incur the cost of performing it.

zero consumption (Chew and Epstein consider paths with strictly positive consumption in every period). The former extension allows the incorporation of the experimental evidence of Keren and Roelofsma (1995) and Weber and Chapman (2005), and the latter is necessary to consider stopping probability (as developed in the main body).

Let  $\mathcal{C}_+ := \{c = (c_0, c_1, \dots, c_t, \dots) : 0 \leq c_t < L < \infty \forall t\}$  represent the set of bounded nonnegative consumption paths. The choice space is  $P := M(\mathcal{C}_+)$ , which is the set of Borel probability measures on  $\mathcal{C}_+$ . Preferences  $\succeq$  over  $P$  are reflexive, transitive, complete, and continuous.<sup>20</sup> A typical element  $p \in P$  is a probability distribution over consumption paths. For any  $n \geq 0$ , denote by  $(c_0, c_1, \dots, c_n, p)$  the consumption path in which consumption until period  $n$  is deterministic, and in which the uncertain consumption levels in later periods are represented by the probability measure  $p$ .

*Certainty additivity* requires that  $\succeq$  agrees with separable additive ordering on deterministic paths.<sup>21</sup>

**Certainty additivity:** The utility of a deterministic consumption path is additively separable. There exist  $u : [0, \infty) \rightarrow (-\infty, \infty)$  such that  $u' > 0$  and  $u'' < 0$ , and  $0 < \beta < 1$  such that the function  $U$  defined by

$$(A1) \quad U(c) := \sum_{t=0}^{\infty} \beta^t u(c_t)$$

represents  $\succeq$  on the subset of  $P$  that includes all deterministic paths.

Note that this representation implies that the “pure” time preference is constant. Hence, when faced with a deterministic consumption path, the agent’s preferences are time consistent (Strotz 1955). This axiom corresponds to the case of  $r = 0$  in (1).

*Homotheticity* implies that preferences between two stochastic consumption paths will not change if all outcomes will be multiplied by a positive constant. For any  $p \in P$  and  $\lambda > 0$ , denote by  $\lambda \circ p \in P$  the measure obtained from  $p$  by scaling all outcomes by the factor  $\lambda$ .

**Homotheticity:** For all  $\lambda > 0$  and  $p, p' \in M(\mathcal{C}_+) : p' \succeq p$  iff  $\lambda \circ p' \succeq \lambda \circ p$ .

The assumption of homotheticity together with continuity of  $\succeq$  on  $P$  implies that  $u(\cdot)$  in (A1) has the form:<sup>22</sup>

$$(A2) \quad u(c) = \frac{c^{1-\alpha}}{1-\alpha} \quad 0 < \alpha < 1.$$

Together with continuity, homotheticity guarantees that the felicity function exhibits CRRA with the coefficient of relative risk aversion smaller than unity. Hence, the elasticity of intertemporal substitution (which equals  $1/\alpha$ ) is greater than one. Although this representation is not general, it allows one to separate risk aversion from intertemporal substitution, and relate to diminishing impatience.

<sup>20</sup> Endow  $\mathcal{C}_+$  with the product topology and let  $M(\mathcal{C}_+)$  be endowed with the weak convergence topology. Note that  $c_t$  is bounded from above: hence, compact completeness and continuity (as in Chew and Epstein 1990) translate here to standard assumptions.

<sup>21</sup> For an example of axiomatization of the latter, see Tjalling C. Koopmans (1960).

<sup>22</sup> Since we require that preference be continuous on paths that include possibly  $c_t = 0$ , the case of  $\alpha \geq 1$ , which is unbounded from below, is excluded. Chew and Epstein’s (1990) Appendix has some treatment of this case imposing limited continuity.

The axiom of *ordinal stochastic dominance* requires that if the distribution of present discounted utility induced by  $p'$ , first order stochastically dominates the distribution of present discounted utility induced by  $p$ , then  $p' \succeq p$ . This axiom implies that the agent prefers consumption paths with lower hazard rate in (1).

**Ordinal (stochastic) dominance:** For every  $p, p' \in P$  : if  $p' \{c \in \mathcal{C}_+ : c \succeq \bar{c}\} \supseteq p \{c \in \mathcal{C}_+ : c \succeq \bar{c}\}$  for all  $\bar{c} \in \mathcal{C}_+$  then  $p' \succeq p$ .

The *stationarity axiom* maintains time consistency if  $c_0$  and  $c_1$  are deterministic, but allows dynamic inconsistency if the consumption level in period 0 or 1 is stochastic. That is, the source of changing preferences could be *only* uncertain present or immediate future consumption: the evaluation of a (random) consumption path at the present could differ from the evaluation of the same consumption path *after* the present or next period's uncertainties have been resolved. In (1), it allows  $g(\cdot)$  to be different from the identity function and allows choices to exhibit preference reversal due to the certainty effect, as in Section I.

**Stationarity:** For all  $c_0, c_1 > 0$  and  $p, p' \in P$  :  $(c_0, c_1, p') \succeq (c_0, c_1, p)$  if and only if  $(c_1, p') \succeq (c_1, p)$ .<sup>23</sup>

For every random consumption path  $\tilde{c} = (\tilde{c}_0, \tilde{c}_1, \dots, \tilde{c}_t, \dots)$  distributed according to  $p$ , let  $\phi(p)$  denote the CDF of the present discounted utility— $\sum_{t=0}^{\infty} \beta^t u(c_t)$ . That is,  $\phi(p) := F_{U(\tilde{c})}$ , where  $\tilde{c}$  is any stochastic consumption path  $\tilde{c} = (\tilde{c}_0, \tilde{c}_1, \dots, \tilde{c}_t, \dots)$  distributed according to  $p$ . Theorem 0 shows that preferences over stochastic consumption paths satisfy the axioms above if and only if they are represented by a certain class of functions that are defined over the CDF of the present discounted utility.

**THEOREM 0 (Chew and Epstein 1990):** *Let  $\succeq$  be a complete, transitive, and continuous preference on  $P$  that satisfies certainty additivity, homotheticity, ordinal dominance, and stationarity. Then, there exists an extended real valued function  $V$  defined on  $\phi(P)$  such that  $V(\phi(p')) \geq V(\phi(p))$  if and only if  $p' \succeq p$ .  $V(\cdot)$  is increasing in the sense of first-order stochastic dominance and satisfies the following functional requirements:*

(a)  $V(F_{\delta_x}) = x$  if  $F_{\delta_x} \in \text{domain } V$ , where  $\delta_x$  is a degenerate probability distribution on  $x \in \mathbb{R}$ , and  $F_{\delta_x}$  is the corresponding CDF;

(b) *Scale invariance:*  $V(F_{\lambda \bar{x}}) = \lambda V(F_{\bar{x}}) \forall \lambda > 0$ ;

(c) *Translation invariance:*  $V(F_{b+\bar{x}}) = b + V(F_{\bar{x}})$  on a suitable domain,  $0 \leq b < \infty$ .

**PROOF:**

For every  $c \geq 0$ , let  $\mathbf{c}_c := (c, c, c, \dots)$  and define  $V$  as follows:

$$(A3) \quad V(\phi(p)) := U(\mathbf{c}_c) \text{ if } p \sim \mathbf{c}_c.$$

<sup>23</sup> Note that if the time line is simply shifted by one period relative to Chew and Epstein (1990), then “stationarity” would imply:  $(c_0, p) \succeq (c_0, p')$  iff  $p \succeq p'$ . That is, it will not allow for time inconsistency as a result of uncertain future (if the present is certain), which is the focus of the current work.

Each  $p$  induces a distribution  $F_{U(\bar{c})}$  on lifetime utility. Ordinal dominance implies that only this distribution is relevant in evaluating consumption paths. From compactness and continuity, for every  $p$  there exists  $c_c$  such that  $p \sim c_c$ . Since  $\alpha < 1$ ,  $c^{1-\alpha}/(1-\alpha) \geq 0$  for  $c \geq 0$ , so  $U(c)$  in (A1) is a real valued function on  $\mathcal{C}_+$ . Hence,  $V(\cdot)$  is well defined. The preference is representable by a function:

$$(A4) \quad U(p) = V(\phi(p)) = V(F_{U(\bar{c})}) = V\left(F_{\sum_{t=0}^{\infty} \beta^t \frac{\bar{c}_t^{1-\alpha}}{1-\alpha}}\right).$$

The rest of the proof imposes restrictions on  $V(\cdot)$ . Ordinal dominance implies that  $V(\cdot)$  is increasing in the sense of first-order stochastic dominance, and this follows from (A3). By taking monotone transformation,  $V$  can be taken to be a certainty equivalent value: hence Property 1 follows. Scale invariance follows from homotheticity and translation invariance follows from stationarity and scale invariance.

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