Diminishing Impatience: Disentangling Time Preference from Uncertain Lifetime

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Abstract

A decision maker with time consistent preferences may exhibit diminishing impatience, when uncertain lifetime is accounted for. Uncertain lifetime captures not only the risk of mortality, but also the possibility that a promise for a delayed reward might be breached, or a postponed consumption might not be realized. The restrictions that time consistency imposes on additive intertemporal preferences are characterized. It is shown that if the hazard rate of mortality is diminishing, then a time consistent agent will exhibit diminishing impatience. A demographic model that allows for unobservable heterogeneity in frailty (risk of mortality) accommodates diminishing impatience, even in the presence of stationarity and time consistency.

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1 Introduction

Uncertain lifetime is a metaphor for situations in which a planned consumption path might not materialize. One cause could be mortality, but other reasons - like breach of promise or exogenous disappearance of future reward, are possible too. This paper studies the effect of uncertain lifetime on time discounting when intertemporal preferences are time additive and satisfy the expected utility assumptions. The importance of uncertain lifetime in temporal choice problems has been acknowledged since Rae ([24], 1834), and has been incorporated into the analysis of optimal consumption and saving problems since Yaari ([39], 1965). However, previous studies that accounted for uncertain lifetime did not have the vast experimental evidence documenting diminishing impatience, which is available today. The focus of this paper is on the apparent tension between time consistency and this evidence.

Following Yaari's work [39], it has been well understood that when uncertain lifetime is accounted for, the risk of mortality and time preference enter symmetrically into the utility function (e.g. Blanchard [5]). An outside observer can only observe time discounting - that is, marginal rates of intertemporal substitution. If uncertainty is present then time discounting is composed of time preference, which stands for intertemporal substitution under full certainty, and odds of realizing consumption.

Time consistency means that when an individual can re-optimize, she does not have an incentive to deviate from her ex-ante plan. Strotz [34] proved that in a deterministic model, time consistency is equivalent to constant time preference. Many experimental studies have tested the descriptive applicability of the time consistency assumption. The leading evidence brought against it is *diminishing impatience*: people are more sensitive to a given time delay in consumption, if it occurs earlier rather than later[1]. That

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[1] For a survey of experimental results and interpretation see Section 4.1 in Frederick et al [9]. Constant time preference suffers from many other anomalies (see Section 4.2 in

is, as two dates are moved uniformly further into the future, the willingness to sacrifice later consumption for an earlier consumption diminishes. This experimental evidence seems to be inconsistent with the functional restriction that time consistency imposes on time preference, and has led many researchers to argue that the intertemporal utility function exhibits decreasing discount rates[2]. Although all of the experiments (I am aware of) were conducted at a single point in time, there seems to be a belief that actual preference reversal would be observed if the preferences were elicited again after some time has passed. To support this view, anecdotal evidence of demand for commitment devices is brought (e.g. footnote 14 in [9]).

This paper characterizes the restrictions that time consistency imposes on additive preferences when lifetime is uncertain. It is shown that if the decision maker is Bayesian, time consistency is equivalent to constant time preference, and does not impose any restrictions on the mortality process. Hence, if preferences are time additive and stationary, satisfy expected utility and time consistency (hence time preference is constant), a decreasing hazard rate over the disappearance of future reward (mortality) translates into diminishing impatience. Casual observation may indicate that many scenarios (especially interactive) satisfy this property: an agent may assign some positive risk that a promise for a reward in a week will not be kept, but *conditioning* on a promise for a reward in a year being kept, the probability that a promised reward in 53 weeks will not be delivered is lower (and may be zero). This explanation is consistent with the experimental findings of Benzion *et al* [4] who found strong support to the hypothesis that delayed consequences have an *implicit risk value* (Rotter [26], Mahrer [21], Mischel and Grusec [22]). It

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Frederick et al [9] for a survey), but the focus of this work is diminishing impatience, since it led the mainstream literature to adopt specific functional forms that can accommodate this evidence. See also Ok and Masatlioglu's [23] discussion of the other anomalies in their Section 4.1 .

[2]This is why this experimental evidence has become to be known as "hyperbolic discounting," as opposed to exponential (or geometric) discounting - where the discount rate is constant.

asserts the individual may assign some positive *subjective* probability (based on past experience) that a delayed reward will not be paid and a delayed payment will not be collected.

However, within the context of lifetime, it is well known that the hazard rate of mortality (after childhood) is increasing and not decreasing. The reminder of the paper is devoted to building a realistic demographic model that may be consistent with diminishing impatience. The model analyzed follows the demographic literature (Vaupel *et al* [37], Vaupel [36]) and incorporates unobservable *frailty* (risk of mortality), and an increasing hazard rate of mortality conditional on frailty . The result is driven by a learning argument: an agent is born with a prior belief over her frailty. As time passes and the individual survives, her conditional expected frailty decreases. If learning is sufficient to cancel the increase in the hazard due to aging, the subjective mixture distribution exhibits decreasing hazard rate, and is consistent with the empirical evidence on diminishing impatience.

Note that the above argument can be made even if the individual does not consciously acknowledge the uncertainty of lifetime as motivating her behaviour. The economic tradition of analyzing decision-theoretic problems requires the environment to be described accurately and the choices to be observed by the modeler. The preferences derived should be consistent with both. Furthermore, even for short horizons, when the probability of actual death is negligible, how can an individual be absolutely certain that a promise for future reward will be kept? Lifetime uncertainty captures this aspect of doubt, which leads to a conservative behaviour.

The current paper is organized as follows. After a short survey of the related literature, Section 2 shows that Strotz's result generalizes to uncertain lifetime. That is, the agent's decisions are time consistent only if time distance is discounted exponentially. Section 3 uncovers the relation between diminishing impatience and the decreasing hazard rate property. Section 4 shows that if individual's frailty is unknown ex-ante, demographic models

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that include belief updating can accommodate decisions that exhibit diminishing impatience. Section 5 concludes. All technical proofs appear in the Appendix.

1.1   Related Literature

In an intriguing paper, Sozou [32] considered a situation where the only component of time discounting is the risk of mortality (no pure time preference), characterized by a constant hazard rate. He showed that if the decision maker has a prior belief over her hazard rate of mortality, she will exhibit diminishing impatience, but time consistent choice. Azfar [2] extended this framework by allowing exponential time preference[3]. These papers relate to the demographic model presented in Section 4, and their technical aspects are discussed in Section 4.2.1. The current paper complements this line of reasoning along two dimensions: theoretical - in characterizing time consistency under uncertain lifetime and disentangling time preference from the risk the prize might disappear or the agent might die[4]; and applied - in suggesting a realistic demographic model that is consistent with diminishing impatience and time consistent choice.

Several other works try to explain diminishing impatience without imposing specific structure (hyperbolic) on time preference. Read [25] argues that diminishing impatience is a result of subadditive discounting and not diminishing time preferences. Subadditive discounting implies that when the interval between two temporal payoffs is divided into subintervals then the total discounting increases. He presents experiments that are consistent with subadditive discounting, while rejecting the predictions of hyperbolic discounting. Rubinstein [28] advocates a procedural approach based on simi-

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[3]Weitzman [38] uses a similar framework to analyze the effect of constant - but unknown - discount rate. As noted by Azfar [2], this framework leads to time inconsistent choices.

[4]That is, after showing that time consistency is independent of the stochastic process governing mortality, a realistic demographic model with an increasing hazard rate could be analyzed.

larity relations (which was developed originally by Rubinstein in the context of choice under risk [27]). According to this approach, when a decision maker compares a consumption of x in time t to consumption of x' in t' she first looks for dominance (more is preferred to less, and sooner is preferred to later); if there is no dominance she looks for a dimension of similarity: is x similar to x' or is t similar to t'? If she views one of the dimensions as similar, then her preferences are determined by the other dimension. If the first two steps are not decisive, then a different criterion is applied. Rubinstein compared the performances of the procedural approach he proposed and hyperbolic discounting in a series of experiments, and showed that the former can explain some choice patterns that are inconsistent with hyperbolic discounting. In an important recent work, Ok and Masatlioglu [23] provide a representation for preferences on the (certain) prize-time space that relies on weakening of transitivity while maintaining separability between disutility of time delay and utility of outcomes. Their representation encompasses, inter alia, exponential discounting (when stationarity and transitivity are imposed), hyperbolic discounting, Read's [25] subadditive discounting and Rubinstein's [28] procedural similarity. Note that in their deterministic framework , diminishing impatience could be accounted for only by relaxing stationarity (hyperbolic discounting) or transitivity (procedural similarity and subadditivity), while the current work maintains both. Furthermore, diminishing impatience implies in their model(s) naive time inconsistency, while here time consistency is maintained.

Fernández-Villaverde and Mukherji [10] present a model where preferences are shocked in every period, but the agent learns about the shock in the current period before consumption decisions are made. They show that in this framework, different agents (who receive different shocks during the current period) may take different decisions when deciding between current consumption and a near future consumption, but as the time horizon is shifted to the future, the current different shocks become irrelevant, and all

agents will make the same choice. Furthermore, they present experimental evidence which shows that the demand for commitment devices is quite limited, contrary to what is conjectured by the literature that hypothesize that preference reversal is a probable consequence of diminishing impatience. Dasgupta and Maskin [8] rationalize hyperbolic discounting in an environment in which payoffs may be realized early[5]. They show that the decision maker becomes more impatient as the horizon is shortened since the likelihood of early realization diminishes in time.

Several recent papers have been motivated by diminishing impatience to model time consistent agents who exert self control cost. Gul and Pesendorfer [14] derive axiomatically a representation of preferences that optimally trades-off the temptation of immediate consumption with the long-run interests of the decision maker. Benhabib and Bisin [3] provide a model in which the agent has access to dynamic commitment (self control) strategies. Fudenberg and Levine [11] model dynamic decisions as a game between a sequence of myopic selves and a long run patient self. The latter can manipulate the utility of the myopic selves (exert self control), and hence achieve its long-run goals. An important distinction between those papers and the current work is that here there is not need to exert self-control cost in order to achieve the long run target: there is simply no conflict between the objectives of the ex-ante self and the interim "selves."

2 Time Consistency with Uncertain Realization

The problem analyzed in this section is a "cake eating" problem, similar to the original problem analyzed by Strotz [34]. The added feature here is that at any point in time the remaining wealth (cake) might disappear (or the agent might die). The goal of this exercise is to characterize the discount

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[5]As an alternative to timing realization they consider waiting cost.

function under which the agent's decisions will be time consistent: she will not have an incentive to deviate in the future from her ex-ante plan.

2.1   The Environment

Consider an allocation problem of an individual with an unknown lifetime. As noted above, uncertain lifetime captures the notion that at any point in time the unconsumed remaining stock might disappear. For simplicity I abstract from all other uncertainties (e.g. income), which could be included in the analysis (see Yaari [39]). The ex-ante optimal program at time 0, and the optimal program conditional on living (the remaining stock has not disappeared) at time $t > 0$ are characterized. The agent's time of death (or the stock's time of disappearance) is denoted by T. Since she does not know her time of death, T is a random variable with pdf $\pi(T)$ on $[0, \infty)$. The probability that the consumer will be alive at time s is given by the survival function $\Omega(s)$:

$$\Omega(s) = \int_s^\infty \pi(t)\, dt \tag{1}$$

The hazard rate at s, which is the pdf of T conditional on the agent living at time s, is given by:

$$r(s) = \frac{\pi(s)}{\Omega(s)} \tag{2}$$

Integrating both sides of (2), shows that the hazard rate fully characterizes the distribution of T by the following relation:

$$\Omega(s) = e^{-\int_0^s r(t)\, dt} \tag{3}$$

Let $\pi_t(s)$ where $t \le s$ denote the conditional pdf of T at s given the consumer is alive at t :

$$\pi_t(s) = \frac{\pi(s)}{\Omega(t)} \tag{4}$$

and let $\Omega_t (s)$ be the probability the consumer will be alive at s conditional on being alive at t :

$$\Omega_t (s) = \int_s^\infty \pi_t (\tau) \, d\tau = \frac{\Omega (s)}{\Omega (t)} \tag{5}$$

The consumer has an endowment of $K (0)$ which she would like to allocate to consumption between 0 and T. Hence, the problem is to find the optimal consumption path. Assume no depreciation, so the law of motion of the state variable K is given by:

$$\frac{dK (s)}{ds} = -C (s) \tag{6}$$

The consumer's intertemporal utility function is additive separable and stationary, when her instantaneous utility function is increasing and concave. She evaluates uncertain prospects using expected utility, and discounts future consumption by the discount function $\alpha (\cdot)$, which is a function of the time distance between the future and the present.

2.2 Optimization

Given this simple environment, the consumer's optimization problem at time t is:

$$\max \int_t^\infty \Omega_t (s) \, \alpha (s - t) \, u (C (s) , s) \, ds \tag{7}$$

$s.t.$

$$\begin{cases} \frac{dK(s)}{ds} = -C (s) \\ K (t) > 0 \; given \\ K (s) \geq 0 \text{ and } C (s) \geq 0 \; \forall \; s \geq t \end{cases}$$

The standard optimality conditions are given by:

$$\Omega_t(s)\,\alpha(s-t)\,u'(C(s),s) = \lambda(s) \ \forall \ s \geq t \tag{8}$$

$$\frac{d\lambda(s)}{ds} = 0 \Longrightarrow \lambda(s) = \widehat{\lambda}_t$$

The ex-ante planning problem is characterized by substituting $t = 0$ into (7), and the optimality conditions are:

$$\Omega(s)\,\alpha(s)\,u'(C(s),s) = \widehat{\lambda}_0 \ \forall \ s \geq 0 \tag{9}$$

That is, the expected discounted marginal utility is constant along the optimal path.

2.3   Time Consistency

I follow Strotz [34] in characterizing the discount function for which the consumer's choices at time t will abide by her original plan. The agent is *naive time consistent*, if she has no incentives to deviate ex-post from her original plan, even when her original plan is naive in the sense that it does not take into account potential disagreements between future and current preferences over consumption paths. The difference from Strotz is that here the consumer does not know her time of death (or when the remaining stock - $K(t)$ - will disappear), hence she might die (or the remaining stock might disappear) before consuming all of $K(0)$, a situation which is not ex-post optimal.

Theorem 1 *A Bayesian decision maker is naive time consistent if and only if she discounts time exponentially, that is:*

$$\alpha(t) = Ae^{-\delta t} \ \ for \ A > 0 \ and \ \delta \in \Re$$

Proof. See Appendix. ∎

Hence, Strotz's [34] result survives uncertain lifetime. The intuition behind the result is that although time preference (characterized by $\alpha(\cdot)$) and survival probabilities (characterized by $\Omega(\cdot)$) enter symmetrically into the utility function, Bayesian updating of $\Omega(\cdot)$ implies that a time consistent agent would need to have constant time preference. If the consumer is impatient, then $\alpha(\cdot)$ is a non-increasing function, and hence $\delta \geq 0$[6]. Note that time consistency says nothing about the mortality (or disappearance of the remaining stock) process. In particular, it does not imply the constant hazard rate property.

3 Diminishing Impatience

From now on assume the agent is time consistent, and therefore $\alpha(t) = Ae^{-\delta t}$. When uncertainty concerning the realization of consumption exists, as in the case of uncertain lifetime, the observed marginal rate at which an individual is willing to substitute utility between two periods (Uzawa [35]) is composed of pure time preferences and belief concerning survival. Following [9] we call this marginal rate of substitution - *time discounting.* Hence, the marginal rate of substitution of utility at t for utility at $t + \tau$, is given by (using (3)):

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[6]Burness [6] generalized Strotz's result, and showed that when the discount factor may depend on both the planning date and the date of consumption, and not only on the time distance between them, naive time consistency is equivalent to a multiplicative exponential function of each argument. In this paper, I impose the restriction that the discount function will be a function of the time distance only, and time affects impatience through the conditional probability of mortality only (which is a function of the time distance too.)

$$MRS_{t,t+\tau} = \frac{e^{-\delta t}\Omega(t)}{e^{-\delta(t+\tau)}\Omega(t+\tau)} = \tag{10}$$

$$= \frac{e^{-\delta t}e^{-\int_0^t r(s)ds}}{e^{-\delta(t+\tau)}e^{-\int_0^{t+\tau} r(s)ds}} = e^{\delta\tau + \int_t^{t+\tau} r(s)ds}$$

Experiments have shown (for a recent survey of results see Frederick, Loewenstein and O'Donoghue [9]) that this function is decreasing in t. That is, the rate at which an individual is willing to substitute utility in t for utility in $t + \tau$ is a decreasing function of t. This is the diminishing impatience phenomenon that is commonly described as "hyperbolic discounting." In the presence of uncertain lifetime, time discounting (MRS) is composed of time preference $(e^{\delta\tau})$ and the inverse of the probability the agent will be alive at $t + \tau$ conditional on being alive at t $\left(\frac{1}{\Omega_t(t+\tau)} = \frac{\Omega(t)}{\Omega(t+\tau)}\right)$. If the agent is time consistent (hence time preference is constant), the evolution of $\Omega(\cdot)$ determines the path of time discounting.

Theorem 2 *If the agent is time consistent then she exhibits diminishing impatience (time discounting) if and only if the uncertainty about lifetime has the decreasing hazard rate property.*

Proof.

$$\frac{dMRS_{t,t+\tau}}{dt} = e^{\delta\tau + \int_t^{t+\tau} r(s)ds}\left[r(t+\tau) - r(t)\right]$$

Hence:

$$sign\left(\frac{dMRS_{t,t+\tau}}{dt}\right) = sign\left(r(t+\tau) - r(t)\right)$$

∎

It might be helpful to have a different view of the previous result. According to (10), the rate of change of time discounting at t, is given by: $\delta + r(t)$.

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Hence, the rate at which current utility could be substituted for future utility is increasing with a diminishing rate if and only if the hazard rate, $r(t)$, is decreasing.

Assume the agent is comparing a reward of 100 dollars today to a promised reward of 110 dollars in a year. Although in the comparison itself there is no explicit uncertainty, all real life decisions involve uncertainty. In the simplest case (presented above), the consumer does not know whether she will be alive in a year. There is some probability she might die beforehand, and will not be able to enjoy the promised future reward. This captures the notion that even if the decision maker lives a year, there is some risk the reward will not be available. This reasoning might be motivated by the common wisdom that further away in time is the promise, the lower is the probability it will be fulfilled. Thus, even if the probability of actual death is negligible, the agent might think she is facing risk on the payment side. The willingness to sacrifice later consumption for an earlier consumption might change when the time horizon changes. In comparing 100 dollars in ten years to 110 dollars in eleven years, the decision maker might make the following argument: "Conditional on surviving ten years, the probability of surviving an extra year is higher than the probability of surviving a year from today." On the dual (disappearance) side: "Conditional on the promise of 100 dollars in ten years being kept, the probability that the promise of 110 dollars in eleven years will be honoured is higher than the prior subjective probability that 110 dollars will actually be paid in a year." This result may be motivated from the matching technology we are faced with in everyday life. For example, if I just met a new acquaintance, the probability I will know her whereabouts in a week is lower than the probability I will know her whereabouts in a year and a week, conditional on knowing her whereabouts in a year.

The explanation that diminishing impatience is the result of a decreasing hazard rate, is consistent with the experimental findings of Benzion *et al*

[4]. In their study they found strong experimental support for the *Implicit Risk Approach* (Mischel and Grusec [22]). This hypothesis is part of Rotter's Social Learning Theory (1954, [26]). Rotter claimed that the potential (i.e. utility) of a behaviour (e.g. choice of immediate or delayed reward) is a function of the expectancy (*subjective* probability) it will lead to a reinforcer (outcome) and the subjective valuation (desirability) of the reinforcer. One interpretation of this theory is that the agent will choose the action with the highest subjective expected utility. This theory emphasizes that individual's choices and preferences (personality) represent the interaction of the person with her environment. Life experience builds up a certain set of subjective beliefs, used in the evaluation of alternative actions. This point of view is closely related to the one which motivated Gilboa and Schmeidler's [12] study of Cased Based Decision Theory. When life experience changes, the evaluation of behaviour can change. As the agent accumulates life experience, the harder (but not impossible) it becomes to adjust it. In particular, the subjective probability (which is based on experience) could be different from the objective probability a consequence will occur. In the context of immediate versus delayed rewards, a person may assign a subjective probability (based on experience) that a promise for a delayed reward will not be kept. Support for this interpretation can be found in Mahrer [21], who showed that strengthening children's trust in the promise maker increases the frequency of them choosing delayed rewards over immediate rewards.

To be more concrete, consider the following example of extreme decreasing hazard. Suppose there are two "types" of promises for future reward: those that are kept, and those that are not kept. The prior probability of the latter type is $r > 0$. The difference in the subject's evaluation of a certain present (time 0) reward and a higher reward in near future (time τ), is composed of the implicit risk (r) and time preference ($\delta\tau$). However, when the dates are moved uniformly into the future (to t and $t + \tau$ respectively), they incur the same risk and therefore their evaluation differs only as a result of the

time preference ($\delta\tau$). In spite of the fact that the agent exhibits diminishing impatience, her decision will be time consistent: her preferences at t between rewards at t and $t + \tau$ will conform to her preferences at time 0. The agent will make the following argument:

> "Since the reward has been offered (at t), I should update my belief over the type of promise made, in a way that will reflect the fact that the promise has been kept. Hence, my posterior belief that the promised reward will not be delivered at $t + \tau$ is updated down to 0, and the difference in the evaluation of a reward at time t and a reward at $t + \tau$ depends only on my time preference - $\delta\tau$."

One might argue that individuals have adapted to this environment, and in their answers to experiments cannot abandon this rule of thumb. Even when the intertemporal decision problem is formulated in terms of certainty, the decision maker *frames* it as one involving risk. This reasoning relies on some inertia which is present in the decision maker's decision process: she cannot adjust her hard-wired decision rule to the environment presented at the experiment.

4 Demographic Model of Uncertain Lifetime

As shown in the previous section, Bayesian time consistent individuals are more sensitive to a given time delay if it occurs earlier rather than later, if and only if the hazard rate of mortality is decreasing. However, it has been long suggested (at least since Gompertz, 1825 [13]) that for an individual above age 14, the hazard rate of death is increasing in age. These demographic models are based on an assumption of a homogeneous population. Following the demographic literature (Vaupel *et al* [37], Vaupel [36]), I argue here that if individuals differ in their *frailty* (force of mortality), and when born

have a prior subjective belief over the frailty component of their hazard rate, allowing Bayesian updating of this prior may lead to a behaviour consistent with a (subjective) decreasing hazard rate, although the actual hazard rate (conditional on the true frailty) may be increasing.

Let (T, Θ) be a bivariate random variable. Frailty is represented by the non-negative random variable - Θ, which represents the consumer's endowment of longevity. The individual holds a prior belief over the distribution of Θ, denoted by the absolutely continuous cdf F and the pdf f. Conditional on $\Theta = \theta$, the pdf of T is given by $\pi(s|\theta)$ and the probability that the individual will be alive at time s is given by $\Omega(s|\theta) = \int_s^\infty \pi(t|\theta) \, dt$. Thus lifetime has a mixture distribution, and the unconditional survival function is given by:

$$\bar{\Omega}(s) = \int \Omega(s|\theta) \, dF(\theta) \tag{11}$$

The following Proposition, shows that Strotz's result survives this extended model of unknown frailty:

Proposition 3 *Let (T, Θ) be a bivariate random variable denoting time of death and endowment of frailty, respectively. Denote by $F(\cdot)$ the absolutely continuous cdf of the prior belief over Θ, and by $\pi(\cdot|\theta)$ the conditional pdf of T given $\Theta = \theta$. If the consumer is Bayesian, then her decisions are naive time consistent if and only if she discounts time exponentially, that is:*

$$\alpha(t) = Ae^{-\delta t} \text{ for } \delta \in \Re \text{ and } A > 0$$

Proof. See Appendix. ■

As before, $\delta > 0$ represents impatience.

I follow the demographic literature (that follows Cox [7]) and assume the hazard rate is multiplicatively dependent on the frailty θ:

$$r(t, \theta) = \theta \rho(t) \tag{12}$$

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where $\rho(t)$ is the component of the hazard rate which is time dependent. In what follows, I will construct an example that assumes a reasonable prior belief over frailty and an increasing hazard rate conditional on frailty. Those define a well-behaved mixture distribution. Conditions for a decreasing hazard rate for the mixture distribution will be derived, and according to Theorem 2 they imply diminishing impatience.

4.1 A Gamma Prior Belief over Frailty

I follow Vaupel, Manton and Stallard [37] and assume frailty at birth is gamma distributed[7] with pdf:

$$f(\theta) = \frac{\gamma^k}{\Gamma(k)} \theta^{k-1} e^{-\gamma\theta} \text{ for } \theta > 0 \tag{13}$$

where γ and k are parameters of the distribution, such that $E(\theta) = \bar{\theta} = \frac{k}{\gamma}$ and $Var(\theta) = \frac{k}{\gamma^2}$. This distribution is chosen because of its analytical tractability and flexibility. The following proposition characterizes the evolution of the conditional frailty, independently of the distribution of the conditional hazard rate.

Proposition 4 *If the prior belief over frailty has a gamma distribution then the posterior frailty conditional on surviving t is:*

$$\theta \,|\{T \geq t\} \sim Gamma(k, \gamma(t))$$

where $\gamma(t) = \gamma + \int_0^t \rho(s)\,ds$

Proof. See Appendix. ∎

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[7] Weitzman [38] uses the gamma distribution to capture the distribution of the discount factor in the population.

The consumer is born and does not know her true frailty. She only has a prior belief over it[8]. As she ages, she learns about her true frailty in the following way: if she had been frail (high θ), the probability of a short lifetime would be high. Therefore, as t increases, the probability that she has a low frailty increases, as summarized by the expected value of the conditional distribution of frailty, which is equal to $\frac{k}{\gamma(t)}$.

4.2 Gompertz's (increasing) Conditional Hazard

Gompertz (1825, [13]) was the first who recognized that a hazard rate which is an exponential function of age, captures the behaviour of human mortality in a substantial portion of the empirical life table. Most of the mortality models used today are adaptations of this observation to deviations from the original Gompertz model at certain age intervals (e.g. old age.) Assume that the time-dependent component of the hazard rate follows Gompertz's [13] rule:

$$\rho(t) = e^{bt} \text{ where } b \geq 0 \tag{14}$$

That is, conditional on frailty, the hazard rate is increasing exponentially. The following proposition shows that even now, if $b\gamma < 1$, the subjective mixture distribution has a decreasing hazard rate and the marginal rate of intertemporal substitution is diminishing in time.

Proposition 5 *Assume that conditional on the frailty value, the hazard rate of death follows Gompertz's law and the prior belief over frailty is gamma(k, γ). If $\gamma b < 1$, then a consumer with time-consistent preferences exhibits diminishing impatience.*

Proof. See Appendix. ∎

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[8]Vaupel [36] analyzes the cross generational correlation in lifespan when frailty is inherited.

To gain intuition of this result, normalize k to 1 (exponential prior). Then the condition is $b < \frac{1}{\gamma} = \bar{\theta}(0)$. Thus, the individual believes ex-ante that her frailty is higher than the rate of change of the hazard rate. Then, as time progresses and she survives, she will update her subjective frailty sufficiently to cause the subjective hazard rate (of the mixture distribution) to decrease.

4.2.1 A Constant Hazard and Related Literature

Although a model that assumes a constant hazard rate cannot be supported empirically, it sheds light on the evolution of conditional frailty, and could be applied easily. Assume frailty is the only component of the hazard rate. That is, the hazard would be independent of age (constant hazard rate) and normalized to 1. Then, according to Proposition 4 the conditional distribution of frailty would be $gamma\,(k, \gamma + t)$. It is easy to show that in this case:

$$\bar{\Omega}_t(s) = \left(\frac{\gamma + t}{\gamma + s}\right)^k \tag{15}$$

Hence, time discounting between t and $t + \tau$ is:

$$MRS_{t,t+\tau} = e^{\delta\tau}\left(1 + \frac{\tau}{\gamma + t}\right)^k \tag{16}$$

which clearly is a decreasing function of t (since $\gamma b = 0$) and has the explicit hyperbolic structure suggested in the Psychological literature.

As noted in the Introduction, Sozou [32] presents examples where there is no time preference (time neutrality), utility is the identity function and the hazard rate is constant but unknown to an animal. He shows that the updated hazard is decreasing in time[9]. Azfar [2] discusses a case of unknown constant hazard, and shows that the apparent discount rate is diminishing,

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[9] Sozou and Seymour [33] study the evolution of time discounting allowing for ageing (diminishing fertility) and unknown constant hazard, and find that time discounting will diminish and then increase as the animal ages.

but decisions are time consistent. The results in this paper show that the conditional hazard could be increasing, as long as the subjective mixture distribution of lifetime has the decreasing hazard rate property. Furthermore, Proposition 3 shows that as long as time discounting is exponential, time consistency will prevail and is independent of the evolution of the hazard rate.

5   Concluding Remarks

This paper has suggested an interpretation that when an agent is time consistent, diminishing impatience may be related to the subjective diminishing hazard rate of discontinuance of future consumption. Beyond casual observation and support from psychological theoretical and experimental studies that justify this relation, it has been demonstrated that diminishing time discounting may be supported by realistic demographic models. The latter allow for unobservable heterogeneity in frailty, about which the agent learns as time passes. Developments of inference methods from surveys of expected longevity, as studied in Hamermesh [15], Smith, Taylor and Sloan [31], Hurd et al [18], and Hurd and McGarry [19] seem to support at least part of the empirical implications of the model presented here. For example, Hamermesh [15], Hurd et al [18] and Hurd and McGarry [19] find (in different surveys) that individuals estimate their probability of survival to a target age quite accurately, but overestimate the conditional probability of surviving to even an older target age given survival to the first target - compared to an average timetable statistics. This finding may be explained by an extensive subjective updating of frailty during those critical years.

It is of interest that a line of reasoning similar to the one explored in the demographic model, has been applied extensively in the labour literature to study duration models. This literature investigates how duration of unemployment data (usually aggregate) could be disaggregated. It has been

observed that aggregate data on duration of unemployment exhibit a decreasing escape (hazard) rate from the unemployment state, while many job search theories predict a constant or an increasing escape rates. Salant [29] and Lancaster [20][10] argued that this could be due to heterogeneity of the population: different unemployed are endowed with a different escape rate parameter, and conditional on this parameter - their escape rate is actually increasing. When the data is being aggregated, those with the high escape rate leave the unemployment pool first, causing the aggregate (average) escape rate to decrease. Identification techniques for such mixed processes have been developed by Heckman [16] and Heckman and Singer [17].

A Proofs

Theorem 1 *A Bayesian decision maker is naive time consistent if and only if she discounts time exponentially, that is:*

$$\alpha\left(t\right) = Ae^{-\delta t} \ \ for \ A > 0 \ and \ \delta \in \Re$$

Proof. The consumption at time s as planned at t should be equal to the original plan for time s consumption:

$$u'\left(c\left(s\right), s\right) = \frac{\widehat{\lambda}_0}{\Omega\left(s\right)\alpha\left(s\right)} = \frac{\widehat{\lambda}_t}{\Omega_t\left(s\right)\alpha\left(s - t\right)}$$

$$\alpha\left(s\right) = \frac{\widehat{\lambda}_0}{\widehat{\lambda}_t} \frac{\Omega_t\left(s\right)}{\Omega\left(s\right)} \alpha\left(s - t\right) \tag{17}$$

$$\frac{\Omega_t\left(s\right)}{\Omega\left(s\right)} = \frac{\Omega\left(s\right)/\Omega\left(t\right)}{\Omega\left(s\right)} = \frac{1}{\Omega\left(t\right)}$$

Letting $a\left(t\right) := \frac{\widehat{\lambda}_0}{\widehat{\lambda}_t}\frac{1}{\Omega(t)}$ and $\tau := s - t$, this is a functional equation in $\alpha\left(\cdot\right)$:

$$\alpha\left(t + \tau\right) = a\left(t\right)\alpha\left(\tau\right)$$

---

[10]The first use of such an argument in employment data is by Silcock [30].

Taking logarithm and setting $f(\cdot) := \ln(\alpha(\cdot))$ and $h(\cdot) := \ln(a(t))$, one gets:

$$f(t+\tau) - f(\tau) = h(t)$$

Setting $\tau = 0$: $h(t) = f(t) - f(0)$ hence:
$f(t+\tau) - f(\tau) = f(t) - f(0)$. Letting $\kappa := -f(0)$:
$f(t+\tau) - f(\tau) - f(t) = \kappa$ or by adding and subtracting κ from the lhs:

$$[f(t+\tau) + \kappa] - [f(\tau) + \kappa] - [f(t) + \kappa] = 0$$

Let $\varphi(\cdot) = f(\cdot) + \kappa$. The following functional equation:

$$\varphi(\tau + \tau) = \varphi(\tau) + \varphi(t) \quad \text{for all } t, \tau \geq 0 \tag{18}$$

is Cauchy's basic equation, which, if $\varphi(\cdot)$ is continuous at a point, is uniquely solved by:

$$\varphi(t) = ct \quad \forall t \geq 0$$

(Aczél [1]). It is immediate from (18) that $\varphi(0) = 0$. Hence $\alpha(t) = e^f(t) = e^\varphi(t) - \kappa = e^c t + \ln(\alpha(0)) = \alpha(0) e^c t$, letting $A = \alpha(0)$:

$$\alpha(t) = Ae^{ct}$$

Letting $\delta = -c$ the theorem is proved. ∎

Proposition 3 *Let (T, Θ) be a bivariate random variable denoting time of death and endowment of frailty, respectively. Denote by $F(\cdot)$ the absolutely continuous cdf of the prior belief over Θ, and by $\pi(\cdot|\theta)$ the conditional pdf of T given $\Theta = \theta$. If the consumer is Bayesian then her decisions are naive time consistent if and only if she discounts time exponentially, that is:*

$$\alpha(t) = Ae^{-\delta t} \text{ for } \delta \in \Re \text{ and } A > 0$$

Proof. This problem has a similar structure to the one studied in Theorem 1. The updated pdf of Θ conditional on surviving to t is given by:

$$f_t(\theta) = f(\theta \,|T \geq t) = \frac{f(\theta)\,\Omega(t|\theta)}{\Omega(t)}$$

Hence, the updated survival function:

$$\Omega_t(s) = \Pr\{T \geq s \,|T \geq t\} = \int \Omega_t(s|\theta)\,dF_t(\theta) =$$

$$= \int \frac{f(\theta)\,\Omega(t|\theta)}{\Omega(t)}\Omega_t(s|\theta) = \frac{1}{\Omega(t)}\int \Omega(s|\theta)\,dF(\theta) = \frac{\Omega(s)}{\Omega(t)}$$

Therefore, the functional equation (17) may be reduced similarly to Cauchy's basic functional equation. ∎

Proposition 4 *If the prior belief over frailty has a gamma distribution, then the posterior frailty conditional on surviving t is:*

$$\theta\,|\{T \geq t\} \sim Gamma\,(k, \gamma(t))$$

where $\gamma(t) = \gamma + \int_0^t \rho(s)\,ds$

Proof. Note that under the multiplicative hazard:

$$\Omega(s|\theta) = e^{-\int_0^s r(t,\theta)dt} = e^{-\int_0^s \theta\rho(t)dt} = e^{-\theta\int_0^s \rho(t)dt} = e^{-\theta H(s)} = [\Omega(s)]^\theta$$

where:

$$H(s) = \int_0^s \rho(t)\,dt$$

Now:

$$f(\theta|T \geq t) = \frac{f(\theta, T \geq t)}{\bar{\Omega}(t)} = \frac{\Omega(t|\theta)\,f(\theta)}{\bar{\Omega}(t)} = \frac{e^{-\theta H(t)}\frac{\gamma^k}{\Gamma(k)}\theta^{k-1}e^{-\gamma\theta}}{\bar{\Omega}(t)}$$

Find $\bar{\Omega}(t)$ by the normalization:

$$\frac{1}{\bar{\Omega}(t)}\int_0^\infty e^{-\theta H(t)}\frac{\gamma^k}{\Gamma(k)}\theta^{k-1}e^{-\gamma\theta}\,d\theta = 1$$

Hence:

$$\bar{\Omega}(t) = \frac{\gamma^k}{\Gamma(k)} \int_0^\infty e^{-\theta(\gamma+H(t))}\theta^{k-1}d\theta = \frac{\gamma^k}{\Gamma(k)}\frac{1}{\gamma(t)^{k-1}}\int_0^\infty e^{-\theta\gamma(t)}[\theta\gamma(t)]^{k-1}d\theta =$$

$$= \frac{\gamma^k}{\Gamma(k)}\frac{1}{\gamma(t)^k}\int_0^\infty e^{-\xi}\xi^{k-1}d\xi = \frac{\gamma^k}{\Gamma(k)}\frac{1}{\gamma(t)^k}\Gamma(k) = \left(\frac{\gamma}{\gamma(t)}\right)^k$$

Substituting back:

$$f(\theta|T \geq t) = \frac{e^{-\theta(\gamma+H(t))}\frac{\gamma^k}{\Gamma(k)}\theta^{k-1}}{\left(\frac{\gamma}{\gamma(t)}\right)^k} = \frac{\gamma(t)^k}{\Gamma(k)}e^{-\theta\gamma(t)}\theta^{k-1}$$

The distribution of frailty conditional on surviving to age t is gamma($k, \gamma(t)$)

∎

Proposition 5 *Assume that conditional on the frailty value, the hazard rate of death follows Gompertz's law and the prior belief over frailty is gamma(k, γ). If $\gamma b < 1$, then a consumer with time-consistent preferences exhibits diminishing impatience.*

Proof. Define the cumulative hazard function $H(t)$ by:

$$H(t) = \int_0^t \rho(t)\,dt = \int_0^t e^{bs}ds = \frac{1}{b}\left[e^{bt} - 1\right]$$

Let $\bar{r}(t)$ denote the expected value of the hazard rate conditional on living at least t, and let $\bar{\theta}(t)$ be the expected value of frailty conditional on living at least t. Then:

$$\bar{r}(t) = \rho(t)\bar{\theta}(t) = e^{bt}\frac{k}{\gamma + H(t)} = \frac{ke^{bt}}{\gamma + \frac{1}{b}\left[e^{bt} - 1\right]}$$

It is easy to see that $\frac{d\bar{r}(t)}{dt} < 0$ if and only if $b\gamma < 1$. Hence, the mixture distribution possesses a diminishing hazard rate and Proposition 2 applies.

∎

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