Difficult Decisions*

Yoram Halevy[†], David Walker-Jones[‡], Lanny Zrill[§]

July 25, 2023

Abstract

We investigate the problem of identifying incomplete preferences in the domain of uncertainty by proposing an incentive-compatible mechanism that bounds the behavior that can be rationalized by very general classes of complete preferences. Hence, choices that do not abide by the bounds indicate that the decision maker cannot rank the alternatives. Data collected from an experiment that implements the proposed mechanism indicates that when choices cannot be rationalized by Subjective Expected Utility they are usually incompatible with general models of complete preferences. Moreover, behavior that is indicative of incomplete preferences is empirically associated with deliberate randomization.

Keywords: Incomplete Preferences, Identification, Elicitation, Choice Under Uncertainty, Deliberate Randomization, Experiment

JEL: C91, D01, D81, D9

*Helpful discussions with Robert Aumann, Maya Bar-Hillel, Itzhak Gilboa, Sergiu Hart, Yusufcan Masatlioglu, Aldo Rustichini, Yuval Salant, Colin Stewart, Tomasz Strzalecki and Michael Woodford are gratefully acknowledged. Seminar participants provided very helpful comments. Billur Gorgulu provided excellent research assistance in carefully proofreading the theoretical results. Yoram Halevy and Lanny Zrill gratefully acknowledge financial support from the Social Sciences and Humanities Research Council of Canada. The experiment was approved by The University of Toronto's Social Sciences, Humanities and Education Research Ethics Board (protocol 41591).

[†]University of Toronto and Hebrew University of Jerusalem, yoram.halevy@utoronto.ca

[‡]University of Surrey, d.walker-jones@surrey.ac.uk

[§]Carleton University, lanny.zrill@carleton.ca

> "Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from the normative viewpoint."
>
> Aumann (1962, p. 2)

1 Introduction

Identifying preferences from choice data is one of the cornerstones of economic research and practice. Yet, the most basic aspect of preferences – the decision maker's (DM) mere ability to compare alternatives – has proven to be the most elusive.

One may question if the DM's inability to rank two alternatives is economically meaningful. Revealed preference theory suggests not, as its focus is on choice behavior, and if one alternative is chosen when the other is available, it reveals to the analyst-observer that the former is "preferred" to the latter. In other words, it may be futile to try and learn about behavior that cannot be expressed through choice, and even if the DM cannot rank a pair of alternatives – if they ultimately choose one of them – the analyst can conclude that they behave "as if" one is preferred to the other.

The goal of the current investigation is to propose a simple and incentive compatible mechanism through which a DM can reveal their inability to compare two alternatives. We propose the mechanism, theoretically characterize the behavior that is indicative of incomplete preferences, and implement the mechanism in a controlled experiment.

Consider a DM who faces a sequence of choices between a fixed bet on an event and a series of objective binary lotteries that have the same potential prize as the bet on the event and probabilities of winning that increase from zero to one. For each choice, the DM has three options: choose to bet on the event, choose the objective lottery, or choose to randomize (with equal chance) over the bet and the objective lottery. This resembles the elicitation of a "probability equivalent" (PE) for the bet, but allows the DM to report any closed and convex set (interval) in [0, 1] by indicating more than one objective lottery at which they prefer to randomize over the bet and the lottery. Importantly, the DM is also asked to make a similar sequence of choices between a bet on the complement event and the same objective lotteries. In addition to the two sequences of comparisons, the DM is also asked to choose a mixture between bets on the event and the complement event, so they may choose to either bet on the event or the complement event with certainty, or to randomize over them. The DM is then paid for one of their choices. The first two tasks reveal to the analyst two intervals of "probability equivalents" for the two bets, while the last task reveals if the DM would like to randomize over the bets.

We proceed by theoretically deriving testable predictions for models of complete preferences for choice under uncertainty that could potentially produce non-singleton intervals, and the predicted association between the intervals and randomization in the third task. As a preliminary step, note that a DM whose beliefs are probabilistic (including Subjective Expected Utility) has a singleton probability equivalent for each bet – known also as their "matching probability" (Baillon & Bleichrodt, 2015; Dimmock, Kouwenberg, & Wakker, 2016) – and the probability equivalents for the bet on the event and its complement must sum up to one. Moreover, this DM will never randomize between the bets, except if they are indifferent. Consider now an ambiguity averse DM. Their preferences are convex, hence they could naturally prefer to randomize (hedge) between the bets in the third task. However, if the DM's preferences are described by the Maxmin Expected Utility Model (Gilboa & Schmeidler, 1989) then they will always report singleton intervals that sum up to less than or equal to one.

Only more general multiple prior models – such as Variational Preferences (VP) (Maccheroni, Marinacci, & Rustichini, 2006), or even more general models such as Uncertainty Averse Preferences (Cerreia-Vioglio, Maccheroni, Marinacci, & Montrucchio, 2011) that also include Smooth Ambiguity Averse Preferences (Klibanoff, Marinacci, & Mukerji, 2005; Denti & Pomatto, 2022) – could potentially rationalize choosing non-singleton intervals. We show, however, that very basic and uncontroversial properties of these preferences (e.g. completeness and transitivity with respect to monotone acts) imply exact restrictions on observed behavior under either "integration" (Baillon, Halevy, & Li, 2022b) or "isolation."[1] For example, the Variational Preferences model implies that the upper bounds of the intervals should sum up to

[1] While isolation is equivalent to assuming that the DM responds to each choice problem as if it is the only problem they face, integration allows the DM to take into account possible interdependencies between choices in different problems.

no more than one when isolation is assumed.

The events we use for bets in the experiment are the relative sizes of two geometric shapes, one of which is always bigger. Subjects' ability to differentiate between them relies on their perception: some comparisons are easy while others are more challenging. We manipulate the difficulty of comparisons by changing the shapes' relative sizes and the number of dimensions the subjects need to compare (e.g. squares or rectangles), which simulate attributes of the alternatives.

The subjects know that one shape is bigger, but may be unsure of which one. Importantly, the subject may be aware of their imperfect perception and express their lack of confidence through the type of intervals they report and their randomization.[2]

The main reason for using shapes as stimuli in the experiment is to achieve a binary "induced value" technique that allows us to exclude some potential explanations for randomization (e.g., Cerreia-Vioglio, Dillenberger, Ortoleva, & Riella, 2019) since any uncertainty the DM has about their payoffs from different monetary payments is irrelevant to their choices as long as they prefer more money. A recent body of economic research also argues that "economists have much to learn from studies of imprecision in people's perception of sensory magnitudes" (Woodford, 2020, p. 2) as there is "reason to believe that reasoning about numerical information often involves imprecise mental representations of a kind directly analogous to those involved in sensory perception" (Khaw et al., 2021, p. 2).

We find that the majority of the subjects make choices that are incompatible with a model that is more general than Variational Preferences (Maccheroni et al., 2006) in more than half of the non-trivial choices they make, and many of them actually make choices that cannot be rationalized even by a much more general model of complete preferences that only assumes transitivity with respect to statewise dominance when isolation is assumed. The failure of the VP model to account for the data can be traced to two stylized features of the data: reported intervals are relatively large[3] and the intervals relate to each other in a way that systematically violates the predictions of the VP model. Moreover, intervals that cannot be rationalized by

[2]The model studied by Khaw, Li, and Woodford (2021) features decision makers that are aware of potential errors in their "encoding" of information and account for these potential errors by optimally "decoding" the information, but such a model does not predict a pervasive desire for randomization.

[3]A non-degenerate interval is reported in 41% of the rounds, and in these rounds the average interval length is more than a quarter.

the VP model lead to more randomization between the bets, suggesting that incompatibility with a general model of complete preferences is associated with deliberate randomization.

We show that our results are not due to confusion about the incentive mechanism, trembles, ambiguity seeking, non-stable preferences, deriving utility from mere randomization, or violation of reduction of compound lotteries. Moreover, subjects tend to spend more time in rounds in which they choose non-degenerate intervals, which indicates a deliberate and thoughtful choice process.

A striking feature of the data is that in most rounds subjects tend to make choices consistent with Subjective Expected Utility (SEU) by choosing probability equivalents that sum to one and not randomizing unless both probability equivalents are one half. When choices are inconsistent with SEU, however, subjects' behavior is very likely to be inconsistent with a much more general model of complete preferences. Moreover, the likelihood of subjects making choices inconsistent with a standard model of complete preferences is increasing in the dimensionality of the shapes compared (e.g. when comparing rectangles subjects are nearly twice as likely to be inconsistent with VP as they are when comparing squares).

Two general lessons can be learned from the current investigation. First, it is possible to identify incomplete preference, even with very mild ancillary assumptions, by considering behavior that all models of complete preferences should exhibit. Second, at least in our setup where uncertainty is over beliefs, the data suggests that deviations from the standard model of SEU cannot be accommodated by much more general models of complete preferences but are instead due to the DM's difficulty in ranking alternatives and can actually be better predicted by a simpler model of incomplete preferences.

The rest of the paper is organized as follows: Section 2 introduces the model and the main axioms, Section 3 contains the main theorems, Section 4 introduces the experiment design, Section 5 provides the main data results, Section 6 is a literature review, and Section 7 concludes. There are also two online appendices: Appendix A houses some additional theory results and proofs and Appendix B features some additional details on the experiment and robustness checks. The supplementary materials have more details on the experiment design including the instructions and quizzes for the experiment, and the discrete versions of the results in Section 3 and Appendix A that are necessary for analysing the data from the experiment.

2 Identification

Consider a set S of **states**, a convex set X of **consequences**, and the set $\mathcal{F}$ of all **(simple) acts**, which are measurable functions from S to X. For every $x \in X$, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. Throughout the paper we consider a binary state space, $S = \{\lambda, \rho\}$, and acts that always result in the DM receiving one of two payments, either $m > 0$, or nothing. We thus, as a minor abuse of notation, let $X = [0, 1]$, which is to say we let $x \in [0, 1]$ denote the constant act that assigns a chance x of winning m as opposed to nothing in both states.[4]

To ease the connection to the experimental implementation, we consider two special acts $L, R \in \mathcal{F}$ (short for "bet on left" and "bet on right"). The acts L and R are **bets on the two possible states**: a correct bet wins the DM a payment of $m > 0$, and nothing otherwise.

As is typical, for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, we define the act $\alpha f + (1-\alpha)g \in \mathcal{F}$ as the act that yields $\alpha f(s) + (1-\alpha)g(s) \in X$ for every $s \in S$. If $f, g \in \mathcal{F}$ and $f(s) > g(s)$ for all $s \in S$, then we say f **statewise dominates** g. We model the DM's **preference** on $\mathcal{F}$ as the binary relation $\succeq$, and denote by $\succ$ and $\sim$ the asymmetric and symmetric parts of $\succeq$, respectively.

2.1 Elicitation Mechanism

The DM faces three questions. In the first two, which we call the **probability equivalent questions**, they are asked to choose between each of a sequence of lotteries (constant acts) and L or R respectively. While the bet remains constant in each of these two questions, the probability of the lottery winning increases from 0 to 1. The DM may respond to each of the two questions using an interval. Let $l_L, u_L \in [0, 1]$ with $l_L \leq u_L$ denote the lower and upper bound reported by the DM (for "probability equivalents") for L, and let $l_R, u_R \in [0, 1]$ with $l_R \leq u_R$ denote the lower and upper bound reported by the DM (for "probability equivalents") for R. If $l_i = u_i$ for $i \in \{L, R\}$ then we say the interval is **degenerate**. If an interval is not degenerate, we say it is **non-degenerate**. The DM is not required to report a non-degenerate interval, and can report a single "probability equivalent" if they desire.

[4] In the supplementary materials we let $X = [0, 100]$ denote the set of percent chances of winning the prize m, so as to better match the specifics of the experiment.

If one of the two probability equivalent questions is selected to determine the DM's payment, a **random lottery** with probability of winning $r \in [0, 1]$ is drawn according to some full support distribution. When r is higher than the upper bound of the interval chosen by the DM for the relevant bet (either L or R), then they receive r – that is, they win m with probability r. If r is smaller than the lower bound chosen by the DM for the relevant bet, their payment is determined by the relevant bet, L or R, and they win m iff the relevant bet is correct about the state of the world. If r falls within the chosen interval, then they receive either the relevant bet or the lottery r, with equal chances.

The third question facing the DM, which we call the **randomization question**, asks them to choose between L and R or to randomize between the two. The DM can randomize between L and R by selecting a number $\alpha \in \{0, 0.01, \ldots, 1\}$ to be their probability of selecting $R \in \mathcal{F}$, if not they select $L \in \mathcal{F}$, and thus if this question is used for payment the DM receives the act $\alpha R + (1 - \alpha)L$.

The proposed mechanism generalizes the BDM (Becker, DeGroot, & Marschak, 1964) mechanism as applied to probability equivalents (Karni, 2009). However, betting on both an event and its complement (as in the work of Baillon, Huang, Selim, and Wakker (2018)), together with the option of allowing the DM to choose an interval of "probability equivalents," permits the analyst to identify behavior that was not available before. Further, in the randomization question the DM can mix between the two bets, and while convex (uncertainty averse) preferences could potentially rationalize randomization and intervals being chosen, the following sections demonstrate that even a very general model of (not necessarily convex) preferences imposes refutable predictions on choices across the three questions.

2.2 Axioms

We are now ready to begin stating our axioms. We begin with an axiom that is specific to the context of the experiment, and then move on to more general axioms.

Axiom 0 (Winning is Preferred (WP)).

For all $x, y \in X$, if $y > x$ then $y \succ x$.

Axiom 1 (Completeness (COM)).

For all $f, g \in \mathcal{F}$, either $f \succeq g$ or $g \succeq f$.

Axiom 2 (Weak Statewise Transitivity (WSTR)).

For all $f,\, g,\, h \in \mathcal{F}$, if h statewise dominates f and $f \succeq g$ then $h \succeq g$, and if h statewise dominates f and $g \succeq h$ then $g \succeq f$.

Axiom 2′ (Statewise Transitivity (STR)).

For all $f,\, g,\, h \in \mathcal{F}$, if h statewise dominates f and $f \succeq g$ then $h \succ g$, and if h statewise dominates f and $g \succeq h$ then $g \succ f$.

STR implies WSTR. COM and STR together imply WP.

Axiom 3 (Convexity (CON)).

For all $f,\, g,\, h \in \mathcal{F}$, if $f,\, h \succeq g$, then for all $\alpha \in (0,\, 1)$: $\alpha f + (1-\alpha)h \succeq g$.[5]

Axiom 4 (Transitivity (TR)).

For all $f,\, g,\, h \in \mathcal{F}$, if $f \succeq g$ and $g \succeq h$, then $f \succeq h$.

Axiom 5 (Weak Certainty Independence (WCI) (Maccheroni et al., 2006)).

For all $f,\, g, \in \mathcal{F}$, if $x,\, y \in X$, then for all $\alpha \in (0,\, 1)$:

$$\alpha f + (1-\alpha)x \succeq \alpha g + (1-\alpha)x \implies \alpha f + (1-\alpha)y \succeq \alpha g + (1-\alpha)y.$$

Axiom 5′ (Certainty Independence (CI) (Gilboa & Schmeidler, 1989)).

For all $f,\, g, \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,\, 1)$:

$$f \succ g \iff \alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x.$$

WCI is implied by COM and CI. WCI is particularly compelling in our environment because our environment is so simple. There is a limited scope for complementarity between acts and constant acts in our setting because of the binary nature of our payoffs.

3 Main Theoretical Results

We are now ready to introduce our main theorems that impose necessary conditions onto DM behavior when their preferences satisfy different subsets of our axioms.[6]

[5] The definition of convexity in Axiom 3 differs from the definition given by some other papers. For a clarification of how this definition of convexity relates to the definition given by, for instance, Cerreia-Vioglio et al. (2011), see Proposition 1 and its proof, or Lemma 56 in their paper.

[6] The discrete versions of the results that match the discrete nature of the sliders in our experiment are presented in supplementary materials.

The following proposition relates our axioms to those of some of the major models of choice under uncertainty (with complete preferences).

Proposition 1. *Given X and S as defined above, and WP:*

(i) COM, WSTR, CON, and TR, are implied by the assumptions of the Uncertainty Averse Preferences (UAP) model (axioms A.1 through A.5 from the work of Cerreia-Vioglio et al. (2011)),

(ii) STR and WCI, and the assumptions of the UAP model, are implied by the Variational Preferences (VP) model (axioms A.1 through A.6 from the work of Maccheroni et al. (2006)),

(iii) COM, STR, and TR, are implied by the Smooth Ambiguity Preferences model (Klibanoff et al., 2005; Denti & Pomatto, 2022),

(iv) CON is implied by the Smooth Ambiguity Averse Preferences (SAAP) model (Klibanoff et al., 2005; Denti & Pomatto, 2022).

Proof. See Appendix A. □

Proposition 1 establishes that, given WP,[7] axioms COM, WSTR, and CON, are together weaker than (i.e., implied by) the UAP model and axioms COM, STR, CON, and WCI, are together weaker than (i.e., implied by) the VP model as both UAP and VP additionally assume TR (and Axiom 6 (continuity) from the supplementary materials), and axioms COM, STR, and CON, are together weaker than (i.e., implied by) SAAP as SAAP additionally assumes TR and indirectly results in Axiom 6 (continuity) being imposed.[8] Further, given WP, all of the axioms in this paper together (COM, STR, CON, TR, CI, and Axiom 6 from the supplementary materials) thus constitute the model of Maxmin Expected Utility (Gilboa & Schmeidler, 1989), as is discussed by Maccheroni et al. (2006).

As is laid out in Table 1, we will refer to preferences that satisfy WP, COM, WSTR, CON, and TR, as the **generalized UAP model**.[9] Preferences that satisfy

[7] Proposition 2 in Appendix A establishes that, in the context of our experiment, WP is a weak assumption to make relative to the quite general UAP model.

[8] Implication of Axiom 6 from the supplementary materials by the Smooth Ambiguity Preference model is established by Lemma 6 from the work of Denti and Pomatto (2022).

[9] In a general enough context the UAP model may not satisfy WP, but if always winning is preferred to always losing, namely $y = 1 \succeq x = 0$, which is an innocuous assumption in our context

Table 1: Relationship Between Axioms and Models Under Isolation

Model:	Axioms:
Generalized UAP	WP, COM, WSTR, CON, and TR
Generalized SAAP	COM, STR, and CON
Generalized VP	COM, STR, CON, and WCI

Preferences that satisfy the axioms in the right column are referred to in the current paper as belonging to the respective *generalized* model. Recall that the preferences studied in the existing literature assume additional structure, hence necessarily satisfy the respective axioms.

COM, STR, and CON, will be referred to as the **generalized SAAP model**.[10] Finally, preferences that satisfy COM, STR, CON, and WCI, will be referred to as the **generalized VP model**.[11]

3.1 Isolation Results

In this subsection we study the implications of the different axioms if the DM makes their choices in "isolation." We say the DM **isolates** the three choice problems in each round if for each problem, there is no alternate choice in the problem that makes the DM weakly better off for every possible random lottery r and makes them strictly better off for a set of random lotteries that has a strictly positive probability of being drawn, if this problem is used for payment. The following theorem considers the testable implications of the most general class of complete and convex preferences, assuming (in addition to completeness and convexity) only that Winning is Preferred and Weak Statewise Transitivity.

Theorem 1. *If the preferences of the DM satisfy WP, COM, WSTR, and CON, and the DM isolates the three choice problems, then their upper bounds u_L and u_R are such that:*

$$\frac{1}{2}\Big(\frac{u_L + u_R}{2} + \frac{1}{2}\Big) \geq \min\{u_L,\, u_R\}.$$

given COM, the UAP model does satisfy WP, as is shown by Proposition 2, and thus we include WP in our generalized UAP model. The results that outline the restrictions on the behavior that can be observed in the experiment according to the generalized UAP model (under isolation) are Theorem 1 and Theorem 6.

[10]The results that outline the restrictions on the behavior that can be observed in the experiment according to the generalized SAAP model (under isolation) are Theorem 1, Theorem 7, and Theorem 8.

[11]The results that outline the restrictions on the behavior that can be observed in the experiment according to the generalized VP model (under isolation) are Theorem 2, Theorem 7, Theorem 8, Theorem 9, and Theorem 10.

Proof. Assume by negation that the inequality does not hold, and notice that this implies $\min\{u_L, u_R\} > 0$, and we will reach a contradiction.

As a first step we show that for $f \in \{L, R\}$ for all small $\epsilon > 0$ there exists $z_f \in [u_f - \epsilon, u_f]$ such that $\frac{1}{2}z_f + \frac{1}{2}f \succeq z_f$. Consider two cases:
If $l_f < u_f$ for $f \in \{L, R\}$, and such z_f does not exist then COM implies that for all random lotteries $r \in [u_f - \epsilon, u_f]$ we have $r \succ \frac{1}{2}r + \frac{1}{2}f$, and thus the DM could do strictly better by reducing their upper bound for f by ϵ since then if a random lottery $r \in (u_f - \epsilon, u_f]$ is drawn, which has a strictly positive probability of happening, the DM would then get r instead of $\frac{1}{2}r + \frac{1}{2}f$, and be strictly better off since $r \succ \frac{1}{2}r + \frac{1}{2}f$.
If $u_f = l_f$ for $f \in \{L, R\}$, and such z_f does not exist then COM implies that for all random lotteries $r \in [u_f - \epsilon, u_f]$ we have $r \succ \frac{1}{2}r + \frac{1}{2}f$, which combined with COM and CON implies that $r \succ f$ (otherwise COM says $f \succeq r$ and $r \succeq r$ so CON says $\frac{1}{2}r + \frac{1}{2}f \succeq r$, contradicting the assumption that such z_f does not exist) and thus $\frac{1}{2}r + \frac{1}{2}f \succeq f$ by CON (since $f \succeq f$ by COM). It follows that the DM could do strictly better off by reducing both u_f and l_f by ϵ as then: they get $\frac{1}{2}r + \frac{1}{2}f$ instead of f, which is weakly better – if the random lottery drawn is $r = u_f - \epsilon$, r instead of f, which is strictly better – if the random lottery is $r \in (u_f - \epsilon, u_f)$ a set that has a strictly positive chance of happening, and r instead of $\frac{1}{2}r + \frac{1}{2}f$, which is strictly better – if the random lottery is $r = u_f$. This concludes the first step.

Since we assumed by negation that:

$$\frac{u_L + u_R}{4} + \frac{1}{4} < \min\{u_L, u_R\},$$

there exists $\epsilon > 0$ such that $\frac{1}{4}(u_L + u_R) + \frac{1}{4} < \min\{u_L, u_R\} - \epsilon$. Pick z_L and z_R as in the first step given ϵ, note that $\frac{1}{4}L(s) + \frac{1}{4}R(s) = \frac{1}{4}$ for all $s \in S$, and that WSTR implies:

$$\frac{1}{2}z_R + \frac{1}{2}R \succeq \min\{z_L, z_R\} \text{ and } \frac{1}{2}z_L + \frac{1}{2}L \succeq \min\{z_L, z_R\}.$$

It follows from CON that

$$\frac{1}{2}\Big(\frac{1}{2}z_R + \frac{1}{2}R\Big) + \frac{1}{2}\Big(\frac{1}{2}z_L + \frac{1}{2}L\Big) = \frac{z_L + z_R}{4} + \frac{1}{4} \succeq \min\{z_L, z_R\},$$

But then WSTR implies that:

$$\frac{u_L + u_R}{4} + \frac{1}{4} \succeq \min\{z_L, z_R\} \text{ and } \frac{u_L + u_R}{4} + \frac{1}{4} \succeq \min\{u_L, u_R\} - \epsilon,$$

which contradicts WP and the fact that $\frac{1}{4}(u_L + u_R) + \frac{1}{4} < \min\{u_L,\, u_R\} - \epsilon$.

$\square$

Behavior that violates Theorem 1 is inconsistent with the generalized VP, SAAP, and UAP models.[12]
We next turn to examine the testable implications of the generalized VP model, which encompasses all known multiple-prior models.[13]

Theorem 2. *If the DM's preferences satisfy WP, COM, CON, and WCI, and the DM isolates the three choice problems, then $u_L + u_R \leq 1$.*

Proof. Assume by negation that $u_L + u_R > 1$, and notice that this implies $\min\{u_R,\, u_L\} > 0$. As shown in the proof of Theorem 1, COM and CON imply that for all small $\epsilon > 0$ there exist $z_L \in [u_L - \epsilon,\, u_L]$ and $z_R \in [u_R - \epsilon,\, u_R]$ such that:

$$\frac{1}{2}z_R + \frac{1}{2}R \succeq z_R \text{ and } \frac{1}{2}z_L + \frac{1}{2}L \succeq z_L.$$

WCI then implies that:

$$\frac{1}{2}z_L + \frac{1}{2}R \succeq \frac{1}{2}z_R + \frac{1}{2}z_L \text{ and } \frac{1}{2}z_R + \frac{1}{2}L \succeq \frac{1}{2}z_L + \frac{1}{2}z_R.$$

and using CON:

$$\frac{1}{2}\Big(\frac{1}{2}z_L + \frac{1}{2}R\Big) + \frac{1}{2}\Big(\frac{1}{2}z_R + \frac{1}{2}L\Big) = \frac{1}{4}z_R + \frac{1}{4}z_L + \frac{1}{4}L + \frac{1}{4}R \succeq \frac{1}{2}z_R + \frac{1}{2}z_L.$$

Note that $\frac{1}{4}L(s) + \frac{1}{4}R(s) = \frac{1}{4}$ for all $s \in S$, so $\frac{1}{4}L + \frac{1}{4}R = \frac{1}{4} \in X$. Thus, if we pick ϵ small enough so that $z_L + z_R > 1$ (which is possible since we assumed that $u_L + u_R > 1$), we contradict WP since:

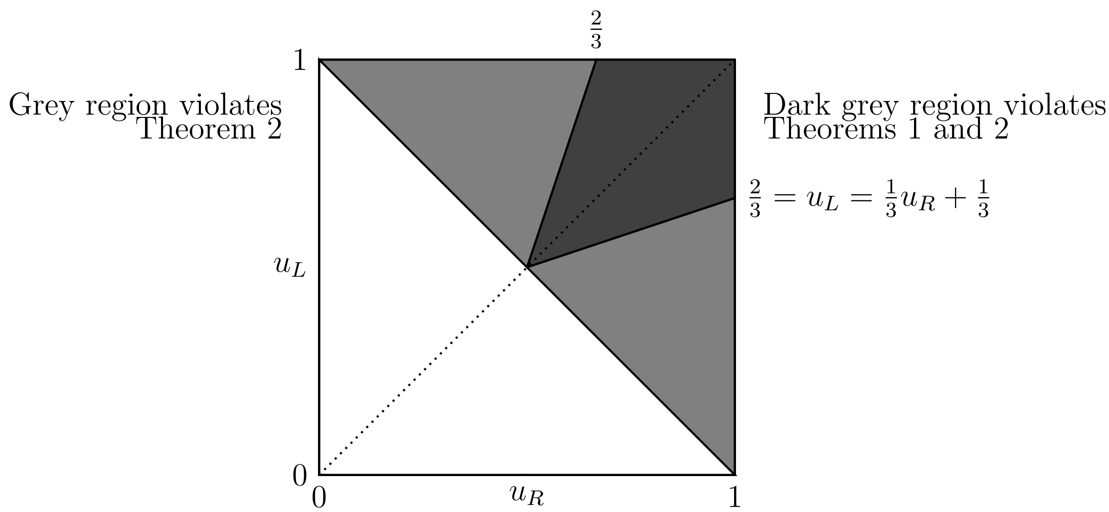
$$\frac{1}{4}z_R + \frac{1}{4}z_L + \frac{1}{4}L + \frac{1}{4}R = \frac{1}{2}\left(\frac{1}{2}z_R + \frac{1}{2}z_L\right) + \frac{1}{2}\left(\frac{1}{2}\right) < \frac{1}{2}z_R + \frac{1}{2}z_L.$$

$\square$

[12] Since STR implies WSTR, while COM and STR together imply WP.

[13] It is easy to show that Maxmin Expected Utility (Gilboa & Schmeidler, 1989) implies the selection of degenerate intervals. Theorem 12 proves it directly from behavioral axioms, and in particular CI.

Figure 1: Testable implications - Theorems 1 and 2



Preferences that satisfy generalized VP (multiple prior) must abide by the restriction that the upper bounds must sum to less than or equal to 1 (white region). Other models that allow for ambiguity aversion (generalized SAAP and UAP) can rationalize the bounds if the upper bounds are far enough from the diagonal (light grey region).

Notice that if $u_L = u_R$, then the upper bounds are consistent with Theorem 2 if and only if they are consistent with Theorem 1. To an extent, this means that if we do not impose WCI and do impose WP, COM, WSTR, and CON, then behavior might be rationalizable if the upper bounds sum to more than one if said bounds are different enough.

Figure 1 depicts the combinations of upper bounds that do and do not violate Theorem 1 and Theorem 2. The white region in Figure 1 contains the pairs of upper bounds that satisfy both Theorem 1 and Theorem 2, the light grey region contains the pairs of upper bounds that satisfy Theorem 1 but violate Theorem 2, while the dark grey region contains the pairs of upper bounds that violate both Theorem 1 and Theorem 2.

Let the **average bound** b_{av} be defined as $b_{av} = \frac{1}{4}(l_L + u_L + l_R + u_R)$. One can think of it (very loosely speaking) as the average of the two probability equivalents, allowing for non-degenerate intervals. Similarly, the **average interval size** s_{av} is defined as $s_{av} = \frac{1}{2}(u_L - l_L + u_R - l_R)$, which is simply the average length of the probability equivalent intervals. The following corollary to Theorem 2 describes the permissible relationship between the two, for a DM whose preferences belong to the generalized VP model.

Corollary 1. *If the DM's preferences satisfy WP, COM, CON, and WCI, then:*

$$b_{av} \leq \frac{1}{2} - \frac{1}{2}s_{av}.$$

Proof. If the DM's preferences satisfy WP, COM, CON, and WCI, then Theorem 2 tell us $u_L + u_R \leq 1$, so:

$$b_{av} = \frac{1}{2}(u_L + u_R) - \frac{1}{2}s_{av} \leq \frac{1}{2} - \frac{1}{2}s_{av}.$$

□

Corollary 1 establishes a monotonic bound on the relation between the two averaged variables for the generalized VP preferences. For example, if preferences are described by the Maxmin Expected Utility Model (Gilboa & Schmeidler, 1989) (and hence satisfy CI) then $s_{av} = 0$ and therefore $b_{av} \leq \frac{1}{2}$. For more general multiple prior preferences, where the probability equivalent intervals might not be degenerate - the bound is even lower.

While the main restrictions on behavior imposed by different general models of complete preferences (based on different subsets of axioms) and the assumption of isolation are found in this subsection, quite a few more related results can be found in Appendix A. The results in Appendix A, Theorems 5 through 11 in particular, also use choices in the randomization question to impose restrictions onto behavior when isolation is assumed.

3.2 Integration Results

In this subsection we study the implications of the axioms if the DM "integrates" when making their choices. First, for each random lottery $r \in [0, 1]$ and each of the three choice questions indexed by $j \in \{1, 2, 3\}$, let $Q_j(r)$ denote the act assigned to the DM when the random lottery drawn is r and question j is used to determine their payment. We then say the DM is **integrating** if they answer a subset of more than one question $\mathcal{Q} \subseteq \{1, 2, 3\}$ in **conjunction**, which means that when the DM answers the subset of questions they have **weights** $\beta_j \geq 0$ for each $j \in \mathcal{Q}$ such that $\sum_{j \in \mathcal{Q}} \beta_j = 1$, and there exists no alternative way of answering the questions that would result in alternate acts $\tilde{Q}_j(r) \in \mathcal{F}$ for each $r \in [0, 1]$ and $j \in \mathcal{Q}$, such that there is an open interval of r with:

$$\sum_{j \in \mathcal{Q}} \beta_j \tilde{Q}_j(r) \succ \sum_{j \in \mathcal{Q}} \beta_j Q_j(r),$$

and $\forall\, r \in [0, 1]$:

$$\sum_{j \in \mathcal{Q}} \beta_j \tilde{Q}_j(r) \succeq \sum_{j \in \mathcal{Q}} \beta_j Q_j(r).$$

Theorem 3. *If the DM's preferences satisfy COM, STR, CON, and WCI, and they consider both probability equivalent questions in conjunction with weights* $\frac{1}{2}$*, then they do not give an interval that is contained in the interior of their other interval, and:* $\max\{u_R, u_L\} + \min\{l_L, l_R\} = 1$, $\min\{u_R, u_L\} \leq \max\{l_L, l_R\}$*, and either* $\min\{u_R, u_L\} = \max\{l_L, l_R\}$ *or* $\min\{u_R, u_L\} + \max\{l_L, l_R\} = 1$.

Proof. See Appendix A. □

It follows that a DM who integrates the two PE problems (with equal weights) and whose preferences are consistent with the generalized VP model must choose intervals that satisfy the constraints in Theorem 3.

Theorem 4. *If the DM's preferences satisfy COM, STR, CON, TR, and CI, and they consider both probability equivalent questions in conjunction with weights* $\frac{1}{2}$*, then* $l_L = u_L$*,* $l_R = u_R$*, and* $u_R + u_L = 1$.

Proof. See Appendix A. □

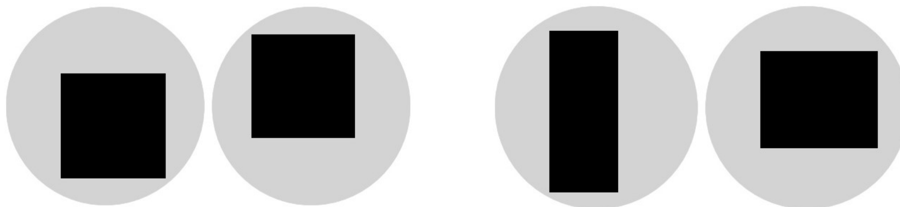
It follows that a DM who integrates the two PE problems (with equal weights) and has preferences represented by the Maxmin Expected Utility Model (Gilboa & Schmeidler, 1989) makes choices as if they are probabilistically sophisticated (see also Baillon, Halevy, and Li (2022a)).

4 Experiment Design

The experiment includes two sections referred to as the "Big-Shape" rounds and "Lottery" rounds, respectively. In each section, subjects are first trained on the interface, before completing a quiz (for which they are rewarded based on performance) that includes five questions – in order to verify their understanding of the

Figure 2: Two examples of figures

(a) Image 2 (b) Image 8



The image on the left is one-dimensional (single-attribute comparison) - comparing two squares, while the image on the right of two rectangles is two-dimensional (multi-attribute comparison).

interface and incentive system. After completing the training and quiz in their first section, subjects complete 12 rounds of decision problems (each round includes two "probability equivalent" questions and the randomization question). Following this, subjects are trained on the other type of round and quizzed again, before completing the other 12 rounds.

At the conclusion of the experiment, one of the three questions in one of the 24 rounds is used to determine if the subject has won a $30 CAD prize.[14] Their payment depends on their choices, a random lottery r, and the resolution of any remaining uncertainty. If a subject ends up winning the prize they receive it in addition to their earnings from the training and quizzes (between an additional $5 and $10 CAD depending on their performance on the quizzes). Payments are made to the subjects using an electronic bank transfer (Interac e-Transfer) within 24 hours of the deadline for each session.

Each Big-Shape round of the experiment involves making comparisons between two shapes of different sizes and judging which of the two has the greater area. See Figure 2 for two examples of such comparisons. Subjects must spend at least 45 seconds in each round before continuing to the next round, but we can observe if they stopped interacting with the interface earlier.

We have four main treatments, with subjects randomly distributed among them in a two-by-two factorial design. The two dimensions we vary are whether the subjects complete the Big-Shape or Lottery rounds first, and which of two different orderings of the questions in each round they see. These treatments are meant to control for

[14]The question that is used for payment is randomly determined before subjects begin the experiment rounds, see the work of Baillon et al. (2022a).

order effects among the different types of rounds and among the different questions, respectively. Additionally, we vary several other factors to control for possible confounds resulting in 64 possible configurations. The construction and rationale for these configurations is described in detail in the supplementary materials.

Subjects were recruited using ORSEE (Greiner, 2015) from the Toronto Experimental Economics Lab subject pool, and thus were essentially all undergraduate students from the University of Toronto from a variety of programs. The experiment was coded with oTree (Chen, Schonger, & Wickens, 2016) and all sessions were done online.

4.1 A Brief Description of the Big-Shape Rounds

The body of the paper focuses on the Big-Shape rounds as they allow us to evaluate if behavior is consistent with general models of complete preferences. Unless stated otherwise, all results refer to the 12 Big-Shape rounds. In each Big-Shape round, subjects respond to the following three questions:

1. What do you think is the chance that the shape in the right circle is larger?
2. What do you think is the chance that the shape in the left circle is larger?
3. Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options?

All three questions are presented to the subject at the same time (though not necessarily in the order above), as well as the image of the shapes,[15] and subjects enter their responses using a graphical user interface. An example of the interface viewed by subjects in the Big-Shape rounds can be found by clicking here.[16]

Subjects enter their responses to the first two questions above, which we call the **probability equivalent questions**, using a double-slider designed to elicit their "probability equivalent(s)" for betting on each shape respectively. Subjects may report a single value or a range of values with the double-sliders, which are incentivized using a discrete implementation of the one analyzed in Section 2.1: before each subject

[15]Exceptionally, in the first Big-Shape round subjects cannot see the image of the shapes. See the supplementary materials for more details on the round with no image of shapes.

[16]The page can also be accessed here: https://teel.economics.utoronto.ca/wp-content/uploads/2023/05/incomplete_interface.htm.

faces the first comparison, a random lottery, r, is drawn from a discrete uniform distribution over $\{0, 1, \ldots, 100\}$.[17] Subjects choose the **upper and lower bounds** for their probability equivalents for each shape, $\{l_i, u_i\}_{i=L,R}$, using the double sliders, with $l_i \leq u_i$ and l_i, $u_i \in \{0, 1, \ldots, 100\}$. If a probability equivalent question is used for payment and $r < l_i$ then the subject bets on the relevant shape, meaning they win the prize if the shape has a greater area than the other shape, while if $r > u_i$ then the subject's payment depends on the lottery r, meaning they win the prize with $r\%$ probability, and, finally, if $r \in [l_i, u_i]$, then the subject has a one-half chance of betting on the shape and a one-half chance of their payment depending on the lottery r. The same incentive scheme is used for the other double-slider that elicits the probability equivalent(s) of betting on the other shape. Figure 6 in Appendix B shows an example of one of the double-sliders, including the dynamically updated text that describes the payoff relevant consequences of the subject's selections. If $l_i = u_i$ for $i \in \{L, R\}$ then we continue to say the interval is **degenerate**, and if an interval is not degenerate it is referred to as **non-degenerate**.

To respond to the third problem, which we call the **randomization question**, subjects use a single slider to set the color composition of an urn that contains 100 balls that are each either red (with an upper case R for 'right') or blue (with an upper case L for 'left'). Suppose they set the slider so that the urn contains $x \in \{0, 1, \ldots, 100\}$ blue balls, and consequently $(100 - x)$ red balls. If the question is used for payment, then the subject bets on the left shape with a $x\%$ chance, meaning they win the prize iff the left shape has the greater area, and bets on the right shape with a $(100 - x)\%$ chance, meaning that they win the prize iff the right shape has the greater area. An example of this particular elicitation device, including the dynamically updated text that describes the payoff relevant consequences of the subject's selections, is given in Figure 7 in Appendix B.

4.2 A Brief Description of the Lottery Rounds

In the section of the experiment that includes the "Lottery" rounds, subjects compare the same shapes as in the Big-Shape rounds, but with different questions. In these rounds, subjects respond to the following three questions:

[17]This number, r, is the same for both probability equivalent questions in all rounds, however, a random lottery is independently drawn for each subject. Subjects are not aware of these details.

1. What percent of the right circle do you think is covered by the shape in the right circle?

2. What percent of the left circle do you think is covered by the shape in the left circle?

3. Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options?

The elicitation mechanism in the Lottery rounds is almost identical to the mechanism in the Big-Shape rounds. The difference in the incentive scheme of the questions is that when a Lottery round question is used for payment a bet on the left or right shape results in the subject winning with a chance that is equal to the proportion of the circle covered by the shape (a proportion that is not known to the participant) instead of winning if the shape is larger (as would be the case if a Big-Shape round question is used for payment instead). For more details on the Lottery rounds see the supplementary materials.

5 Experimental Results

This section employs the discrete versions of our identification results, which can be found in the supplementary materials.[18] Thus, if in this section we say that the data is inconsistent with Theorem Y we really mean that it is inconsistent with the discrete version of Theorem Y. It is important to remember throughout that if a behavior is consistent with continuous data then is is also consistent with discrete data. As a result, it is more demanding to identify a behavior that is inconsistent with a model based on discrete data.

This section only uses data from the 12 "Big-Shape" rounds (other than in the last subsection). The dataset that we use for analysis contains the choices of the 218 subjects that completed the experiment, and our analysis in the current section assumes that subjects are "isolating."

[18]If the continuous version of a result is called Theorem Y in this paper, then the discrete version is called Theorem Y.1. The same is true of Corollary 1.

5.1 Consistency of Responses in a Round

This subsection documents that subjects' responses in the three problems in each round are based on a consistent appraisal of the choice problem, and thus it seems particularly innocuous to interpret their choices as deliberate and representing their ranking and beliefs.

If a subject chooses a non-degenerate interval for the left shape there is a 97% chance they also choose a non-degenerate interval for the right shape, and vice versa. This extreme correlation is not being driven by the subjects overwhelmingly reporting intervals either, far from it in fact: overall, there is only a 41% chance of a non-degenerate interval being chosen in a round.

If choices are consistent with Subjective Expected Utility with a stable appraisal of a choice problem then we should expect that two degenerate intervals were reported and that the probability equivalents for either shape being larger should sum up to 100, and this is what we see in 96% of rounds where two degenerate intervals are given, despite the fact that common models of ambiguity aversion, such as Maxmin Expected Utility (Gilboa & Schmeidler, 1989), would predict that subjects could report probability equivalents that sum up to less than 100% even if preferences are stable (assuming the DM isolates the three choice problems). Since subjects can report non-degenerate intervals in our experiment it seems that an appropriate and analogous measure of consistency in our setting would be to check if lower bounds sum up to 100 or less, and upper bounds sum up to 100 or more, though, again, many models of ambiguity aversion would predict this is not the case even if preferences are stable. In 97% of rounds the lower bounds sum up to 100 or less and in 97% of rounds the upper bounds sum up to 100 or more.

The consistency of responses to the two probability equivalent problems, and the fact that subjects simultaneously responded to all problems in a round,[19] seem to indicate that responses reflect deliberate stable preferences, rather than a noisy measure thereof.

[19] We say subjects answered the problems in a round simultaneously because they were presented with all three problems on the same page and had the ability to change their choices to any of the three problems until they moved onto the next round.

Table 2: Aggregate Distribution of Behavior

	Percent of rounds
Consistent with SEU	53%
Non-degenerate interval(s) of PE but inconsistent with generalized VP	37%
Non-degenerate interval(s) of PE consistent with generalized VP	4%
Two degenerate intervals that sum to more than 100	1%

Classification of consistency with the generalized VP model is based on Theorem 2 – considering responses in the two probability equivalent questions, and Theorem 7, Theorem 8, Theorem 9, and Theorem 10 – considering also the response in the randomization question.

5.2 Descriptive Statistics

A substantial proportion (53%) of the Big-Shape rounds feature observed behavior that is consistent with Subjective Expected Utility (SEU): two degenerate intervals are reported, the probability equivalents chosen for the left and right shapes sum up to 100, and if the probability equivalents are different from 50 then in the randomization question the subject assigned a 100% chance to betting on the shape they assign a higher probability equivalent to be larger.

Choices made in only 4% of rounds feature at least one non-degenerate interval of probability equivalents and are consistent with the generalized VP model: the pair of intervals satisfies Theorem 2, and the response to the randomization question is consistent with Theorem 7, Theorem 8, Theorem 9, and Theorem 10.

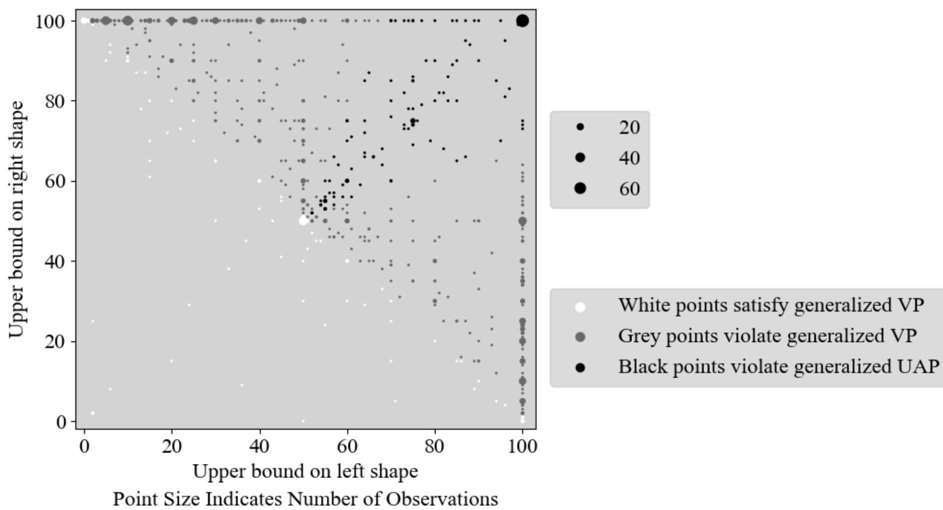
In 37% of the rounds subjects chose at least one non-degenerate interval of probability equivalents, yet were inconsistent with the generalized VP model. This behavior suggests incomplete ranking.

In only 1% of the rounds subjects chose two degenerate intervals for probability equivalents that sum to 101 or more (which may be rationalized by ambiguity seeking behavior).[20]

Figure 3 displays the combinations of upper bounds that are observed in the 41% of rounds that feature a non-degenerate interval, and is the empirical analogue of Figure 1. In Figure 3, the white points indicate observed combinations of upper bounds that satisfy both Theorem 1 and Theorem 2, and hence are consistent with

[20]The remaining 5% of rounds that are not accounted for in Table 2 do not fit nicely into any of the four categories described in Table 2. For instance, in some rounds two different degenerate intervals are reported and the upper bounds sum to 100 yet in the randomization question the subject did not assign a 100% chance to betting on the shape to which they gave a higher probability equivalent to.

Figure 3: Upper Bounds: Empirical distribution of consistency with Generalized VP and UAP preferences in rounds with a non-degenerate interval



White points are consistent with generalized VP preferences. Grey points are inconsistent with generalized VP but could be rationalized by UAP preferences. Black points are inconsistent with UAP preferences.

generalized VP preferences. The grey points indicate observed combinations of upper bounds that satisfy Theorem 1 but violate Theorem 2, and thus are inconsistent with the generalized VP model, but are consistent with generalized UAP preferences. The black points indicate observed combinations of upper bounds that violate both Theorem 1 and Theorem 2, and thus are inconsistent with generalized VP, SAAP, and UAP preferences. What is evident from Figure 3 is that, when a non-degenerate interval is reported, behavior is usually inconsistent with general models of complete preferences (such as VP, SAAP or UAP, models).

Individual analysis that counts the proportion of subjects that are inconsistent with flexible models of complete preferences in the Big-Shape rounds, reveals it to be quite high. Table 3 describes the number of subjects that violate different combinations of the axioms. In particular, the different rows in Table 3 provide information on how many subjects violated the generalized VP model, generalized SAAP model, or generalized UAP model, and how many violated the generalized VP model when convexity (CON) is relaxed. Moreover, as a robustness check, we exclude the round with no image and the rounds with easy images (Images 5 and 11 in Figure 9 in the supplementary materials) for each subject, and find that 51% of subjects still violate

Table 3: Individual analysis: number of subjects inconsistent with alternative models

Row #: 1	Number of subjects (NOS)	218
2	NOS inconsistent with generalized VP model	183 (84%)
3	NOS inconsistent with generalized VP model 3 or more times	144 (66%)
4	NOS inconsistent with generalized VP model 5 or more times	119 (55%)
5	NOS inconsistent with generalized VP model without CON	99 (45%)
6	NOS inconsistent with generalized SAAP	159 (73%)
7	NOS inconsistent with generalized SAAP 3 or more times	92 (42%)
8	NOS inconsistent with generalized UAP model	81 (37%)

Table 4: Likelihood of Randomization (LOR) Conditional on PEs responses

LOR over L and R in a round	63%
LOR if two different degenerate PEs	13%
LOR if two identical degenerate PEs	98%
LOR if a non-degenerate PE interval	91%
LOR if PEs violate Theorem 1	95%
LOR if PEs violate Theorem 2	92%

The table documents the likelihood of a subject choosing to randomize between L and R, unconditionally and conditionally on their choices in the probability equivalents questions. The last two rows depict the LOR conditionally on PEs that are inconsistent with generalized UAP and VP models.

the predictions of the generalized VP model in at least five of their nine remaining Big-Shape rounds (this result is not displayed in Table 3).

If the data is consistent with a model of complete preferences that features a set of acts where preferences are uncertainty averse (convex) and another set of acts where preferences are uncertainty seeking, then it seems logical that we would observe more behavior that appears to be uncertainty (ambiguity) seeking in Table 2. Thus, convexity seems to be a natural property to impose on preferences, yet it leads to substantial rejections of quite flexible models of complete preferences, as is demonstrated by Table 3, though, as row 5 in Table 3 indicates, assuming convexity (CON) is not essential to reject the predictions of generalized VP for a substantial proportion of subjects.[21] Further, as Table 4 indicates, when subjects violate Theorem 1 and Theorem 2 it is extremely likely that they also choose to randomize over L and R in the randomization question.

[21] Theorem 5 and Theorem 11 together only impose COM, STR, and WCI, not CON. Row 5 in Table 3 thus discusses the number of subjects that violated Theorem 5 or Theorem 11 at least once.

The rows in Table 3 make only minimal assumptions about the structure of preferences, with only winning is prefered (WP), weak statewise transitivity (WSTR), and completeness (COM) assumed throughout. Nevertheless, we observe that a significant proportion of subjects make choices that are inconsistent with these assumptions. Hence, if we accept WP and WSTR as particularly innocuous, then these inconsistency may be taken as evidence against the more demanding assumption of completeness (COM). Moreover, in Section 5.3 and Section 5.4 we demonstrate that a simpler model of incomplete preferences does a better job on average than generalized VP at predicting the relationship between the intervals in rounds with a non-degenerate interval. The rest of this subsection argues that this conclusion is robust to a number of other potential explanations.

It is not, for instance, lack of understanding on the part of subjects that is driving our estimates of incomplete preferences either. As is discussed in Appendix B.6, quiz performance has essentially no impact on the chance of a subject violating general models of complete preferences. During the quiz, subjects are not able to move on from a question until they answer it correctly, so they are in a sense forced to learn the correct answer, and in each round they are provided with dynamic text that explains the implications of their choices (see Figure 6 and Figure 7). It thus seems that the forced learning combined with the dynamic text explanations meant that even subjects that did not initially retain as much of the information from the training largely interpreted the mechanisms in the same way as those that performed quite well on the quiz, and thus seem to have understood it quite well. This claim that subjects largely understood the mechanisms in the rounds is substantiated by the consistency of the data that is discussed in the previous subsection.

Analysis of response time data also supports the conclusion that apparent incompleteness of preferences is not being driven by subjects who are rushing through the experiment and are not being careful, as subjects who made choices inconsistent with Theorem 2 (generalized VP) spent more time adjusting their sliders. Detailed reaction time data that records the exact time at which they made any adjustment of any slider is available for 215 of our 218 subjects:[22] when subjects made choices inconsistent with Theorem 2 in their PE questions in a round, it took them on average 73 seconds to finish with their sliders, with a median time of 44 seconds; while

[22]Reaction time data for two subjects was lost due to a server error, and there are errors in one of the subject's reaction time data that seems to be due to a local error on their device.

if subjects made PE choices consistent with Theorem 2 in these questions it took on average 50 seconds[23] to reach a final decision using the sliders, with a median time of only 31 seconds.

In some models of stochastic choice a subject could strictly prefer to randomize over a set of identical options (e.g., Fudenberg, Iijima, & Strzalecki, 2015), and would thus randomize for the sake of randomization. However, we believe that these models do not provide a good alternative explanation for our data since in 59% of rounds subjects choose two degenerate intervals in the probability equivalent questions, 94% of subjects have at least one round where they choose two degenerate intervals in the probability equivalent questions, and 95% of subjects have at least one round where they chose either L or R in the randomization question. It thus does not seem to be the case that subjects prefer to randomize for the sake of randomization. The implications of the work of Fudenberg et al. (2015) in the context of the sanity check round with no image are discussed in the supplementary materials, and analysis on subjects that passed all three sanity checks can also be found in Appendix B.5.

Our main experimental results are robust to a number of other potential concerns as is discussed in Appendix B. For instance, Appendix B.3 addresses problems with reduction of compound lotteries and "integration" (Baillon et al., 2022b) across questions. Appendix B.4 also explores whether the data could have been generated by "trembling hands," and increase the rejection threshold from Theorem 2 to 105 and even 110. Increasing the rejection threshold along these lines has very little impact on how prevalent rejections of Theorem 2 are (as can be surmised from Figure 3).

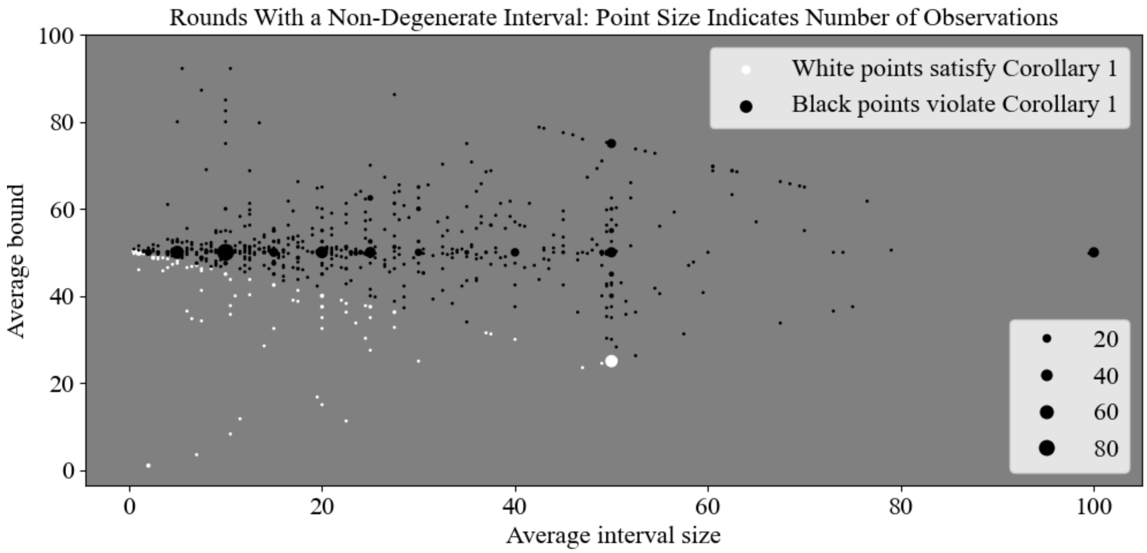
5.3 Systematic Errors in the Interval Predictions of the Variational Model

Not only do generalized Variational Preferences (VP) fail to account for a substantial proportion of the choice data, but they are actually systematically incorrect when predicting the relationship between the two intervals within a round. We believe this is one of the most striking features of our data.

Recall that the average bound in a round – b_{av}, is the average of the two bounds on the probability equivalents for both the left shape and the right shape. Given

[23]This is statistically significantly less time on average according to a two-sided t-test at the 1% level.

Figure 4: Empirical distribution of consistency with generalized VP preferences as a function of average bound and average interval size



Probability equivalent intervals that are consistent with generalized VP preference are depicted in white, while those that are inconsistent with generalized VP are depicted in black.

Theorem 2, it follows that for choices to be consistent with the generalized VP model, the average bound in a round with a non-degenerate interval cannot be more than $\frac{1}{2}$[24] minus half of the average size of the of two intervals in the round – s_{av}, as is shown by Corollary 1. Subjects choose a non-degenerate interval in 41% of the rounds. However, in 87% of these rounds the average bound is inconsistent with the predictions of Corollary 1.

The white dots in Figure 4 illustrate the number of rounds in which subjects have a non-degenerate interval and a combination of average bound (average of their four bounds l_L, u_L, l_R, and u_R) and average interval size (average of the difference between l_L and u_L and the difference between l_R, u_R) that do not violate Corollary 1. The black dots in Figure 4 illustrate the number of rounds in which subjects have a non-degenerate interval and a combination of average bound and average interval size that violate Corollary 1. As the average interval size increases, it is clear from the white dots that the maximal average bound that is acceptable according to Corollary 1 decreases, but, as is evident from Figure 4 and further illustrated in the next subsection, increases in the average interval size do not seem to be associated

[24]In the discrete setting relevant to the experiment the relevant number is instead 50.

with a reduction in the average bound in the data. In rounds with a non-degenerate interval: 71% of average bounds are between 45 and 55 and 51% of average bounds are between 49 and 51.

5.4 Predicting Intervals with Bewley

Bewley (2002) proposed one of the seminal theories of incomplete preferences, using a model of Knightian uncertainty in which the DM's preference incompleteness stems from their uncertainty about their beliefs concerning the likelihood of states of the world. Application of Bewley's model in our context seems natural as it would be reasonable to assume that subjects may lack confidence in their beliefs about which shape is larger. More recent models of incomplete preferences allow for a "dual" incompleteness due to "tastes" (as opposed to incompleteness due to beliefs as in Knightian uncertainty). For instance, Ok, Ortoleva, and Riella (2012) propose a model in which a DM is either unsure of their beliefs or their tastes, but not both, while Galaabaatar and Karni (2013) allow for a DM to be unsure of both their beliefs and their tastes. In our context, where only two possible prizes are possible, it seems that uncertainty about "tastes" should be irrelevant as long as the agent knows they strictly prefer winning the prize to not winning it.

In the work of Bewley (2002) the DM has a convex set of probabilities over states of the world, and one act is preferred to another iff it is ranked higher according to every probability in the set. Thus, while both Bewley's model and the Maxmin Expected Utility Model (Gilboa & Schmeidler, 1989) feature sets of prior beliefs, the role of the prior beliefs in the two models differ, and a bridge between the two models is provided by Gilboa, Maccheroni, Marinacci, and Schmeidler (2010).

Bewley (2002) further assumes that there exists a "status quo" option, and if the DM cannot directly compare two options - they simply select the status quo option. As there is no apparent "status quo" in our setting, we amend this model and assume that the DM chooses to randomize over options when they are not comparable. Given this assumption, we can obtain simple predictions for the interval reported by a DM for one shape based on their observed interval for the other shape.

Suppose the DM reports an interval $[a, b]$, with $0 \leq a < b \leq 100$, when asked for "probability equivalents" of one shape being larger than a second shape. If we assume that the DM randomizes iff they do not have strict preference over the two options, then $[a, b]$ is their set of probabilities (percentage chances) of the one shape being larger according to our amended version of the Bewley (2002) model. Thus, their set of probabilities (percentage chances) of the second shape being larger than the first shape is $[100 - b, 100 - a]$, and we can predict that they would report this interval of "probability equivalents" for the other shape being larger (ignoring discreteness issues for now). We call this the **Bewley prediction** for probability equivalents. Interestingly, this prediction is very different than what would be predicted by our generalization of the Variational Preference (VP) model when assuming the DM isolates the three choice problems. Theorem 2 imposes that the upper bound chosen for the other shape (which has not had its reported interval observed yet) is less than or equal to $100 - b$ (see Theorem 2, ignoring discreteness issues for now), and thus under isolation the generalized VP model predicts a set of intervals instead of a specific interval, and the single interval that is the Bewley prediction is not in the set of predicted intervals as long as $a < b$ (a non-degenerate interval is observed).

As is shown in Section 5.3, these set predictions of the generalized VP model under isolation do quite poorly, but Bewley predictions, as can be surmised from Figure 4, actually perform extremely well on average, and, as predicted, the horizontal line at 50 is highly populated. If we look at rounds where subjects report a non-degenerate interval for the left shape then the average error of the Bewley predictions for the lower and upper bounds for the right shape are -1.26 and 1.38 respectively, and if we look at rounds where subjects report a non-degenerate interval for the right shape then the average error of the Bewley predictions for the lower and upper bounds for the left shape are 0.04 and 1.01 respectively.[25] These average errors are tiny, and thus the Bewley predictions are quite accurate on average. Further, as can be observed from Figure 4, the errors do not seem to be bi-modal.

Some readers may have noticed that the Bewley predictions are similar to the predictions made by our generalization of the VP model under integration (see Theorem 3 for the result in the continuous case), but there is one key distinction. Under integration, our generalization of the VP model predicts that the intervals of "probability equivalents" reported by the DM should not overlap (no part of either interval

[25]In rounds with a non-degenerate interval for the left shape the average absolute value of the Bewley prediction error for the lower and upper bounds for the right shape are 11.64 and 10.90 respectively, and in rounds with a non-degenerate interval for the right shape the average absolute value of the Bewley prediction error for the lower and upper bounds for the left shape are 10.23 and 12.17 respectively.

should be contained in the interior of the other interval, see Theorem 3). In contrast, if the DM reports an interval $[a, b]$ with $a < 50 < b$ for one shape, then the Bewley prediction would say that we should expect an overlap between the DM's two intervals of "probability equivalents."

The Bewley predictions do much better than the generalized VP model under integration in this regard. If a subject reports a non-degenerate interval for the left (right) shape and the amended Bewley model predicts an overlap between the two intervals, we find an overlap 78% (82%) of the time in our data. If a subject reports a non-degenerate interval for the left (right) shape and the amended Bewley model predicts no overlap between the intervals we find an overlap 15% (14%) of the time.

The characterization of the Bewley model (with weak preference) provided in Gilboa et al. (2010) in conjunction with our generalized VP axioms allows us to further focus on which axiom should be relaxed to account for the choice patterns we observe. Assuming WP, the generalized VP model is implied by Bewley model axioms in Gilboa et al. (2010) if completeness is assumed (as independence and transitivity from the work of Gilboa et al. (2010) imply convexity (CON)). It follows that the axioms underlying the generalized VP model could be strengthened without leading to further violations, as long as completeness is relaxed, but requiring completeness leads to inconsistencies.

On the completeness axiom, von Neumann and Morgenstern wrote: "It is very dubious, whether the idealization of reality which treats this postulate as a valid one, is appropriate or even convenient" (von Neumann & Morgenstern, 2007, p. 630). It seems that our setting is one in which it is more accurate to model subjects as having potentially incomplete preferences.

5.5 Squares of Interest vs Rectangles of Interest

There are two sets of choice problems that are particularly helpful for understanding the circumstances in which one should expect behavior that rejects the generalized VP and SAAP models, and hence suggests incompleteness of preferences. Each subject faces a round with two rectangles (Image 8 in Figure 2), which we call the **rectangles of interest**, and a round with two squares (either Image 2 in Figure 2 or Image 2B in Figure 9 in the supplementary materials), which we call the **squares of interest**. Importantly, the areas of the two squares in Image 2 are identical to the areas of the rectangles of interest (but the side with the bigger shape is switched) and the areas of the two squares in Image 2B are essentially identical to the areas of the squares in Image 2 as can be observed in Figure 9 in the supplementary materials.[26]

The goal of these comparisons is to study if and how subjects' choices vary when they confront choice problems that are similar in terms of how challenging it is to identify the larger shape, but differ in terms of "dimensionality" – when sides' length are the relevant attributes of the choice objects. It follows that squares have a single attribute, while rectangles have two attributes. We take the chance that a subject bets on the larger shape in the randomization question as a suitable proxy for how challenging it is to identify the larger shape. The 105 subjects (call them Group 1) that see Image 2B and Image 8 have an average chance of betting correctly of 54% when faced with both the squares and rectangles of interest, and the 113 subjects (call them Group 2) that see Image 2 and Image 8 have average chances of betting correctly of 62% when faced with the squares of interest and 56% when faced with the rectangles of interest, which is not a statistically significant difference.[27] We conclude that identifying the larger shape when faced with the squares and rectangles of interest is similarly challenging.

Both groups, however, are much more likely to make choices that are inconsistent with the generalized VP model when it is assumed the DM isolates the three choice problems (Theorem 2 – that focuses only on the PE questions, Theorem 7, Theorem 8, Theorem 9, or Theorem 10 – that incorporate the randomization question) when faced with the rectangles of interest than when faced with the squares of interest. While the proportion of subjects that is not consistent with the generalized VP model (assuming the DM isolates the three choice problems) when faced with squares of interest is about 30% for both groups, it increases to about 50% when each group is faced with the rectangles of interest, and for both groups this difference is statistically significant at the 1% level according to two-sided Fisher's exact test. Ignoring the rounds with the 2 easy images (Image 5 and Image 11 in Figure 9 in the supplementary materials), the lowest chance of violating the generalized VP model (assuming the DM isolates the three choice problems) is when faced with the squares of interest, at

[26]The squares in Image 2 take up 35.45% and 34.63% of their circles respectively while the squares in Image 2B take up 34.95% and 34.38% of their circles respectively.

[27]The differences in the average chances of betting correctly for the squares and rectangles of interest are not statistically significantly different at the 10% level for either group according to two paired t-tests.

least 6 percentage points lower than any other image they faced for both groups. The probability of the same kind of violations when faced with the rectangles of interest is close to the highest chance (at most 7 percentage point less than the maximal image for both groups).[28] Unsurprisingly, the PE intervals chosen for Image 3 in Figure 9 in the supplementary materials have the highest chance of being inconsistent with the generalized VP preferences.[29]

If we instead look for violations of the generalized SAAP model when it is assumed the DM isolates the three choice problems (Theorem 1, Theorem 7, or Theorem 8) for the squares and rectangles of interest then for both Group 1 and Group 2 it is more likely that the generalized SAAP model is rejected when facing the rectangles of interest as opposed to the squares of interest,[30] but the difference is not statistically significant unless we aggregate across the two groups, at which point the chance of violations for the squares versus rectangles of interest is statistically significantly different at the 5% level according to two-sided Fisher's exact test.

Figure 8 demonstrates the markedly different aggregate behavior for the upper and lower bounds for bets on the right shape in rounds in which the subjects choose between bets on the squares of interest versus rounds when the choice is between bets on the rectangles of interest. When faced with the squares of interest subjects frequently put all 5 sliders to 50 to indicate their indifference between the squares, whereas when faced with the rectangles of interest they are more likely to report a non-degenerate interval. When faced with choice between bets on the squares of interest subjects report a non-degenerate interval 32% of the time, while reporting a non-degenerate interval 54%[31] of the time when faced with the rectangles of interest, even though, as we argued above, how challenging it is to identify the larger shape is

[28]The two groups have the images in the rounds that immediately proceed their respective rounds with squares and rectangles of interest swapped, so it is not differences in the image(s) seen before these rounds that is driving differences in behavior. Further, which of the squares and rectangles of interest is seen first is randomly determined for each subject in each group, so it is not order effects that are driving the differences (see the supplementary materials for more details).

[29]The chance of betting on the larger shape was only 44%, and 57% of subjects made choices inconsistent with generalized VP.

[30]Group 1 violates the generalized SAAP model 18% of the time when faced with squares of interest and 27% of the time when faced with the rectangles of interest, while Group 2 violates the generalized SAAP model 19% of the time when faced with squares of interest and 29% of the time when faced with the rectangles of interest.

[31]This is a statistically significantly higher chance according to a two-sided Fisher's exact test at the 1% level.

essentially the same for the squares and rectangles of interest.[32]

5.6 Incomplete Preferences Result in Randomization

As is established by Table 5, which describes the distribution of behavior in rounds where the subject did not select degenerate PEs of 50, behavior in a round that is not consistent with flexible models of complete preferences is highly correlated with the subject choosing to randomize over L and R in the round.

Table 5: Association of Randomization and Violations, Conditional on Not 50-50

	Randomization over L and R	No randomization over L and R
Consistent with generalized VP	8% (157)	41% (853)
Inconsistent with generalized VP	46% (953)	5% (110)
Consistent with generalized SAAP	30% (616)	44% (915)
Inconsistent with generalized SAAP	24% (494)	2% (48)

Total of 2073 observations. Inconsistency with either generalized VP or SAAP models is strongly associated with randomization over L and R, p-value of two-sided Fisher exact test smaller than 1% for both generalized models.

We can, however, actually go beyond association, as Figure 5 shows, and argue that there is a causal relationship between inconsistency of responses to the PE questions with generalized VP (violations of Theorem 2) and the choice to randomize over L and R.

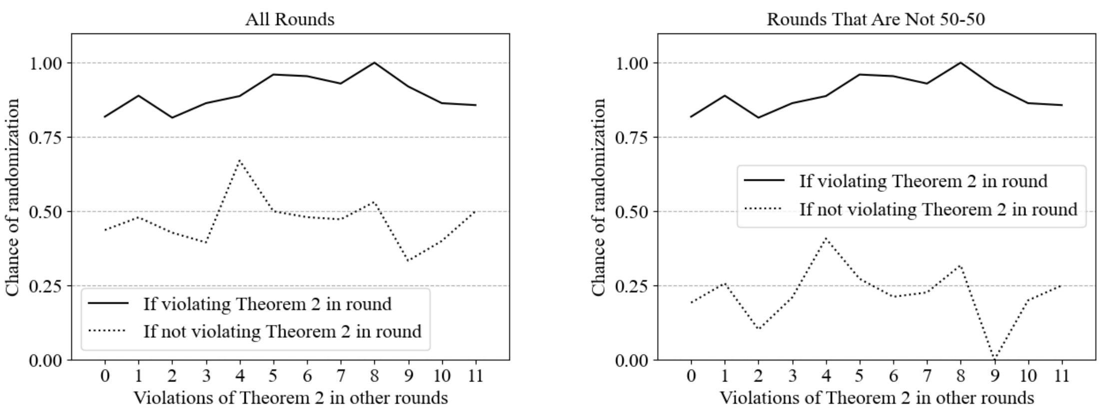
The left-hand side of Figure 5 depicts the chances of randomization over L and R in the randomization question in a specific round on the vertical axis: in solid black – if the responses to the PE questions are not consistent with generalized VP (violate Theorem 2) in the specific round, and in dotted black – if the responses to the PE questions are consistent with generalized VP (do not violate Theorem 2), while controlling (on the horizontal axis) for the number of responses to the PE questions in the other rounds the subject has that are inconsistent with generalized VP. For each number of inconsistencies with generalized VP (violations of Theorem 2) in the other rounds, the chance of randomization is substantially higher if the subject is inconsistent with generalized VP (violates Theorem 2) in the specific round. What is even more surprising is that the frequency of inconsistencies with generalized VP in the other rounds is essentially inconsequential for the chance of randomization in

[32]See Online Appendix B.1 for further discussion.

a specific round. In other words, the chance that a subject chooses to randomize is determined by their choices in the "probability equivalent" questions within a round, but is unaffected by their responses to these questions in the other rounds.

The right side of Figure 5 conducts the same exercise but only uses data from rounds in which the subject did not have both of their probability equivalents degenerate at 50, which we take as a sign of indifference. The right side of Figure 5 accentuates the relationship even further.

Figure 5: Randomization and Inconsistency with generalized VP



The figure depicts the chance of randomizing between L and R in a round controlling for consistency of the responses to the PE questions with generalized VP (Theorem 2) in a round, and consistency in other rounds. A subject is more likely to randomize if their responses to the PE questions in the specific round are not consistent with generalized VP (Theorem 2), but the chance of randomization is independent of such inconsistency with generalized VP in other rounds. Both when including and not including the rounds in which both probability equivalents are degenerate at 50, the chance of randomization when responses to the PE questions are inconsistent with generalized VP is statistically significantly higher than when they are consistent with generalized VP at the 1% significance level according to two-sided Fisher exact test for each number of violations of Theorem 2 in the other rounds except for 11.

5.7 Lottery Rounds

The Lottery rounds are similar to the Big-Shape rounds except for two differences. First, L or R results in the subject winning the prize with a chance that is equal to the proportion of the circle covered by the shape. Second, instead of asking what is the chance of each shape being larger, we ask what percent of the circle they believe is covered by each shape.

The goal of the Big-Shape rounds is to test the behavior for consistency with general models of complete preferences, which is possible to accomplish in a rigorous way because the two probability equivalent questions are for complementary events. Since each shape in the Lottery rounds represents a lottery, it is significantly harder to rule out complementarity between acts and convexity of preferences.

The Lottery rounds are, however, more similar to the type of decision problems that have featured in previous experiments. A typical experiment features choices between two or more options that have values that are not perfectly negatively correlated: all options can have high or low realized values simultaneously, unlike the bets on shapes in the Big-Shape rounds where one bet being beneficial means the other is not. As such, these rounds provide a natural bridge between the stark experimental environment in the Big-Shape rounds and previous experiments.

For the prevalent reporting of non-degenerate intervals in the Big-Shape rounds to be of interest to the broader field of economics it needs to be established that the non-degenerate intervals are not a consequence of the complementarity of the events on which the bets are defined, which is exactly the goal of the Lottery rounds.

We find that subjects are more likely to choose a non-degenerate interval in the Lottery rounds compared to the Big-Shape rounds, reporting a non-degenerate interval in 87% of the former. The intervals reported in the Lottery rounds are smaller on average, but still large, with an average size of 18 when a non-degenerate interval is reported, and this difference in average size makes sense given the differences in the expected incentives. Thus, the main conclusion that should be drawn from the Lottery rounds is that subjects' frequent desire to report intervals in the Big-Shape rounds is not being driven by the perfect negative correlation between the value of L and the value of R that is, for our purposes, a crucial feature of the Big-Shape rounds. The Lottery rounds are discussed in greater detail in the supplementary materials.

6 Literature Review

There have been several previous attempts to experimentally identify incomplete preferences in an incentivized fashion, but we believe that our identification strategy, which relies on "reasonable" choice patterns that are inconsistent with a general class of complete preference (with only minor ancillary assumptions), is unique in several important ways.

Agranov and Ortoleva (in press), for example, elicit ranges of certainty and lottery equivalents using a Multiple Price List (MPL) approach.[33] In one of their treatments, subjects choose between: a lottery with two possible prizes, a sure payment with a third value, or to randomize over the two possibilities (in increments of 10 probability points). While Agranov and Ortoleva (in press) find many instances of ranges for certainty equivalents, they acknowledge that it is impossible with their data to distinguish between various explanations involving complete and convex preferences and incomplete preferences. The induced value nature of our data set, on the other hand, allows us to achieve this crucial distinction.

Cettolin and Riedl (2019) use a similar methodology in one part of their experiment where subjects select from each of 21 pairs of options. Each pair features the same uncertain (ambiguous) option and one of a monotonically ranked set of risky lotteries. As in our experiment, the subjects have the ability to randomize (mix) with a 50-50 chance over the options in each pair. In the second part of their experiment subjects are presented with each of their choices in sequence and their willingness to pay to maintain their original choice is elicited, whether it was the uncertain option, the risky option, or randomizing between the two, rather than be assigned either the uncertain or risky option.[34] The authors differentiate between complete and incomplete preferences by making the following identifying assumption: "If repeated choice of option mix is attributable to a preference for randomization between risky and ambiguous prospects participants should be willing to pay a positive price for keeping mix, whereas subjects should not be willing to pay if incomplete preferences are the reason for choosing mix repeatedly" (Cettolin & Riedl, 2019, p. 549). This identification strategy requires that the preferences of the subject are stable between the first and second part of the experiment, a problem we avoid by eliciting all relevant choice information simultaneously. More crucially, to make this argument they rely heavily on Bewley (2002), by assuming that subjects view the option that they receive if they are not willing to pay to mix to be the "default" or "status quo" option, but if mix is instead taken as the default - as it is what the subject already selected,

[33] Cubitt, Navarro-Martinez, and Starmer (2015) are, to the best of our knowledge, the first to employ this mechanism in order to elicit ranges of values or beliefs. In their study these choices were not incentivized and the focus of their analysis was on the consequences of imprecise preferences rather than identifying incompleteness.

[34] Subjects were randomly assigned to treatments with either the uncertain or certain option as the "default" if they chose to mix.

the identification strategy is problematic.

In another experiment, conducted by Costa-Gomes, Cueva, Gerasimou, and Tejiščák (2022), subjects are asked to pick from 26 menus with 2 to 5 headsets. In one treatment, subjects are forced to make a single selection from each choice set, where as in another treatment they may defer their choice to a later time for a small fee. Gerasimou (2021) conducts a similar experiment but allows subjects to choose non-singleton subsets from 50 menus of up to four pairs of £10 gift cards. In both, they find that choices in the forced choice treatment are less consistent in terms of revealed preference than choices in the unforced treatment. Moreover, they find that a model of incomplete preferences due to Gerasimou (2018) provides a better fit than the standard model of rational choice for most subjects. Taken together, this suggests that subjects may choose to defer choices when they have difficulty ranking the alternatives. That said, Gerasimou (2021) and Costa-Gomes et al. (2022) cannot necessarily rule out many alternative explanations for their findings. For example, subjects not choosing any option, and thus delaying their decision for a small cost, are viewed by these papers as having incomplete preferences, an argument that also requires the stability of preferences. This assumption is questionable as Costa-Gomes et al. (2022) themselves find some evidence of introspective preference learning during the main phase of their experiment. Moreover, models of a preference for flexibility (Kreps, 1979) or costly learning could similarly explain this behavior with complete preferences.[35] Further, their design excludes the possibility that deferral could result from behavior implied by some of the models of complete and convex preferences studied in the current paper. For example, costly choice deferral of this form can be rationalized by the model of Variational Preferences that predicts that subjects may be willing to pay to delay their decision if they strictly prefer to randomize over all options in a particular fashion that is not initially available.

Danan and Ziegelmeyer (2006) are, to the best of our knowledge, the first authors to experimentally try to identify incomplete preferences, and also show theoretically how incompleteness may be identified from a preference for flexibility within

[35]A model of costly learning can rationalize paying to defer a decision when preferences are nevertheless complete due to the particular features of the random incentive system: instead of repeatedly paying the opportunity cost of learning about which option is likely best for the subject, a subject could pay to defer when the optimal decision is not evident, saving the cost of learning on all but at most one of the decisions that are more costly to make, since at most one randomly selected question is used for payment.

a static framework. In their experiment, using somewhat of a hybrid of Agranov and Ortoleva (in press) and Costa-Gomes et al. (2022), the authors elicit a range of certainty equivalents for a given lottery by allowing subjects to defer their choices to a later time at a small cost. Like Gerasimou (2021) and Costa-Gomes et al. (2022), introducing a time element similarly induces a joint test of completeness and stability that may confound their conclusions.

Another recent attempt at identifying incompleteness is made by Nielsen and Rigotti (2022). To the best of our knowledge, they are the first to avoid tying their hands to a specific assumption regarding how subjects with incomplete preferences will choose when forced to, and instead allow subjects to not make some comparisons. This is done by allowing subjects to train an algorithm to estimate their preferences based on their choices. The estimated preferences are then used to make the only directly payoff relevant choice for the subject. While not strictly incentive compatible, the authors argue that their mechanism is nevertheless *behaviorally* incentive compatible and claim that choosing "not to choose" identifies incomplete preferences. Nevertheless, their identification is potentially hampered by subjects' inability to train the algorithm to randomize, so it is possible that subjects choose not to choose when randomization is strictly preferred to both options. Moreover (even ignoring the complexity of the algorithm that is not transparent to the subject), it is not clear what conceptually "estimating preference" means when preferences are incomplete.

There are many other experimental studies that aim to better understand the relationship between so-called "imprecise" preferences and, possibly stochastic, choice behavior. For example, Butler (2000) seeks to explain how the difficulty of determining the ranking between objects may explain observed preference reversals. He quantifies this difficulty using an unincentivized strength-of-preference indicator for recording responses to binary lottery choice problems and shows that more preference reversals occur when preferences are "weak." This is consistent with our hypothesis that incomplete preferences, i.e. those for which the ranking is unclear, may lead the DM to randomize. Butler and Loomes (2007) use a similar strength of preference indicator embedded in a multiple price list — also unincentivized — to elicit ranges of certainty and "probability" equivalents for which subjects are unsure about their preference in comparison to two reference lotteries.[36] They then record the choices of

[36]Butler and Loomes (2007) use a 4 valued strength of preference indicator. The certainty and "probability" equivalents are taken to be the values where subjects switch from weakly preferring

the subjects among the two reference lotteries for three repetitions and compare these to the rankings implied by the elicited preferences in the first stage. This is similar in spirit to our design where preference information (beliefs in our case) is elicited for each choice object separately in addition to observing the (possibly stochastic) choice between the two objects. In a more recent study, Enke and Graeber (in press) investigate how imprecise preferences driven by "cognitive uncertainty" may correlate with the classic S-curve that frequently characterizes estimated probability weighting functions. As in previous studies, the subject's precision of preferences is not incentivized, and it is not clear what "cognitive uncertainty" means in term of choice behavior.

There is also a burgeoning literature that, like the current study, seeks to understand how the difficulty of perceiving the differences between options may impact choice behavior, but many of these papers model the DM as rationally choosing what costly pieces of information to acquire.[37] There is a large theory literature that explores the behavioral implications of different types of cost functions for information (e.g., Matějka & McKay, 2015; Caplin & Dean, 2015; Caplin, Dean, & Leahy, 2022, 2019), and a growing literature that experimentally studies the accuracy of different models of costly learning for predicting DMs' behavior (e.g., Dean & Neligh, in press; Dewan & Neligh, 2020; Denti, 2022). In addition to its shared focus on environments where options can be ranked objectively but are difficult to differentiate between, the costly learning literature also models the choice behavior of a DM as stochastic. At first glance, this all makes the costly learning literature seem particularly pertinent to our paper, and as Butler (2000) argues, incompleteness may arise due to the cost of acquiring information or an inherent difficulty distinguishing between objects.

There is a crucial difference between the data presented in the current paper and the way stochasticity is modelled in the costly learning literature, however. In the latter, the behavior of a DM is typically stochastic because most models predict that the DM will not acquire all of the available information, and what option they select depends on the realized outcome of their information gathering strategy, which

but being unsure about A to weakly preferring but being unsure about B. There are, in a sense, ranges of certainty and "probability" equivalents as subjects reported ranges where they were unsure about their preferences.

[37]There is an older related literature that dates back at least to Tversky (1972) who writes: "Choice probabilities ... reflect not only the utilities of the alternatives in question, but also the difficulty of comparing them." This implies that any useful descriptive theory of choice must account for the potential difficulty of comparing objects.

is stochastic. That said, models in the costly learning literature essentially all agree that if the DM is selecting between binary outcomes and is acquiring any information then, with probability one, when they are done learning they should have a strict preference for picking one of the options outright as opposed to randomizing over the options (Matějka & McKay, 2015; Caplin & Dean, 2015; Caplin et al., 2022, 2019; Denti, 2022). This is in stark contrast with what we see in our experiment where subjects wish to randomize over L and R in the majority of rounds. Moreover, it is easy to show that models of costly learning would predict that a DM should not use the double-sliders to report a non-degenerate interval, which is again in stark contrast with the data we report here.

A similar dichotomy exists between our data and the model studied by Karni (2023) (and Karni (2022)). In the work of Karni (2023), the preferences of the DM are incomplete, and their behavior is stochastic to an outside observer, but the DM does not wish to randomize over options. When the DM cannot directly compare two alternatives their choice is "triggered by impulses" that are inherently random, but result in the DM choosing a single option, not explicit randomization.

Our study is also related to some of the experimental literature on stochastic choice. Sopher and Narramore (2000), for example, use a similar mechanism to the one that we use to allow subjects to randomize over L and R (see Figure 7), but in their setting they allow subjects to select a mixture between two lotteries in order to evaluate various stochastic choice models including the Random Utility model (RUM) and a model of Deliberate Randomization due to Machina (1985). Agranov and Ortoleva (2017) provide further support for deliberate randomization using a repeated choice framework, where stochastic choice is more often observed for comparisons between lotteries deemed "hard."[38] Chew, Miao, Shen, and Zhong (2022) also find evidence in favor of deliberate randomization expressed as multiple-switching behavior in MPLs, though, in their case, subjects would have had to rely on an internal randomization device. Like Sopher and Narramore (2000), Feldman and Rehbeck (2022) also allow subjects to choose mixtures between two three-outcome lotteries, and compare these to repeated choices among the same lotteries. They interpret a positive correlation between these two objects as evidence of deliberate

[38]Dwenger, Kübler, and Weizsäcker (2018) also find a preference for randomization that seems to arise from the difficulty of ranking objects of choice due to uncertainty regarding one's own preferences.

randomization and find more randomization among lotteries for which the odds-ratio is intermediate in value. Hence, even in these studies, which were focused primarily on stochastic choice, the results suggest an important, though not identifiable, connection between the difficulty of comparing objects and stochastic choice.

Finally, our paper is, of course, related to the theoretical literature on incomplete preferences. The main difference with our paper is that instead of directly modelling behavior when preferences are incomplete (e.g., Bewley, 2002; Karni, 2023), or studying axiomatic foundations and mathematical properties for incomplete preferences (e.g., Aumann, 1962; Ghirardato, Maccheroni, & Marinacci, 2004; Nau, 2006; Gilboa et al., 2010; Ok et al., 2012; Galaabaatar & Karni, 2013; Faro, 2015; Riella, 2015), our paper explores the testable implications for the most general models of complete preferences in an enriched dataset, and then empirically characterizes choice behavior when general models of complete preferences are rejected. For a particularly good example of a situation where incomplete preferences seem to make intuitive sense see the introduction of the work of Eliaz and Ok (2006). For another paper that is interested in the potential relationship between stochasticity of choices and the incompleteness of preferences, but uses a different set of assumptions and observations, and, further, produces welfare orderings, see the work of Ok and Tserenjigmid (2022).

7 Conclusion

The assumption that decision makers can always rank all alternatives seems to many scholars normatively and descriptively unappealing. Yet, the revealed preference approach that has been the workhorse of Economic research has made this assumption especially challenging to evaluate empirically. The current work proposes a methodology to evaluate this assumption using an incentivized mechanism in which choices of decision makers with complete monotone preference would be set-identified. Choices that fall outside of this set are indicative of incomplete preference.

We apply this methodology to the domain of choice under uncertainty, in which very general models of complete preference have been developed to accommodate ambiguity-sensitive behavior. We let subjects choose between bets and elicit their probability equivalents, allowing them to express an interval of such equivalents. We also investigate subjects' desire to randomize and if and how it relates to incompleteness. We find that although in most decision problems subjects' choices are consistent with Subjective Expected Utility, in most other cases – in which subjects choose intervals of probability equivalents – their choices are incompatible with complete monotone preferences. Moreover, when choices are indicative of incomplete preferences – decision makers tend to randomize.

Much of behavioral economics has been devoted to studying environments in which decisions are difficult and choices are inconsistent with a "standard" model of preferences. The response in face of such evidence has been to propose more general models of complete preference (ever since the St. Petersburg paradox and Daniel Bernoulli's resolution of it using logarithmic utility). We hypothesize that much of the experimental evidence emerges in situations where decisions are difficult, and agents find it challenging to rank the alternatives they face. Exactly in this twilight zone, decision makers may rely on procedures, algorithms, and heuristics to make choices. For example, in the present investigation, a decision maker who could not rank a bet and a lottery chose to randomize between the two. Once we view the randomization as a procedure the decision maker employs to reach a decision, it becomes natural to expect that in other environments they may use the same or other similar procedures. The behavioral (complete) preferences that have been proposed to rationalize these difficult decisions may have captured some of the procedures' properties, but ignore the context in which they are applied by applying them universally. Naturally, more work is needed to extend the current paper's methodology to other domains, such as choice over time, interpersonal, and interactive, decisions. We believe that it promises to provide a new perspective on behavioral economics by connecting to models of bounded and procedural rationality on which decision makers rely in making difficult decisions.

References

Agranov, M., & Ortoleva, P. (2017). Stochastic choice and preferences for randomization. *Journal of Political Economy*, *125*(1), 40–68.

Agranov, M., & Ortoleva, P. (in press). Ranges of preferences and randomization. *Review of Economics and Statistics*.

Aumann, R. J. (1962). Utility theory without the completeness axiom. *Econometrica*, *30*(3), 445–462.

Baillon, A., & Bleichrodt, H. (2015). Testing ambiguity models through the measurement of probabilities for gains and losses. *American Economic Journal: Microeconomics*, *7*(2), 77-100.

Baillon, A., Halevy, Y., & Li, C. (2022a). Experimental elicitation of ambiguity attitude using the random incentive system. *Experimental Economics*, *25*(3), 1002–1023.

Baillon, A., Halevy, Y., & Li, C. (2022b). Randomize at your own risk: on the observability of ambiguity aversion. *Econometrica*, *90*(3), 1085–1107.

Baillon, A., Huang, Z., Selim, A., & Wakker, P. P. (2018). Measuring ambiguity attitudes for all (natural) events. *Econometrica*, *86*(5), 1839–1858.

Becker, G. M., DeGroot, M. H., & Marschak, J. (1964). Measuring utility by a single-response sequential method. *Behavioral science*, *9*(3), 226–232.

Bewley, T. F. (2002). Knightian decision theory. part i. *Decisions in Economics and Finance*, *25*(2), 79–110.

Butler, D. J. (2000). Do non-expected utility choice patterns spring from hazy preferences? an experimental study of choice errors. *Journal of Economic Behavior & Organization*, *41*(3), 277–297.

Butler, D. J., & Loomes, G. C. (2007). Imprecision as an account of the preference reversal phenomenon. *American Economic Review*, *97*(1), 277–297.

Caplin, A., & Dean, M. (2015). Revealed preference, rational inattention, and costly information acquisition. *American Economic Review*, *105*(7), 2183–2203.

Caplin, A., Dean, M., & Leahy, J. (2019). Rational inattention, optimal consideration sets, and stochastic choice. *The Review of Economic Studies*, *86*(3), 1061–1094.

Caplin, A., Dean, M., & Leahy, J. (2022). Rationally inattentive behavior: characterizing and generalizing shannon entropy. *Journal of Political Economy*, *130*(6), 1676–1715. doi: https://doi.org/10.1086/719276

Cerreia-Vioglio, S., Dillenberger, D., Ortoleva, P., & Riella, G. (2019). Deliberately stochastic. *American Economic Review*, *109*(7), 2425–45.

Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., & Montrucchio, L. (2011). Uncertainty averse preferences. *Journal of Economic Theory*, *146*(4), 1275–1330.

Cettolin, E., & Riedl, A. (2019). Revealed preferences under uncertainty: Incomplete preferences and preferences for randomization. *Journal of Economic Theory*, *181*, 547–585.

Chen, D. L., Schonger, M., & Wickens, C. (2016). otree—an open-source platform for laboratory, online, and field experiments. *Journal of Behavioral and Experimental Finance*, *9*, 88–97.

Chew, S. H., Miao, B., Shen, Q., & Zhong, S. (2022). Multiple-switching behavior in choice-list elicitation of risk preference. *Journal of Economic Theory*, *204*, 105510.

Costa-Gomes, M. A., Cueva, C., Gerasimou, G., & Tejiščák, M. (2022). Choice, deferral, and consistency. *Quantitative Economics*, *13*(3), 1297–1318.

Cubitt, R. P., Navarro-Martinez, D., & Starmer, C. (2015). On preference imprecision. *Journal of Risk and Uncertainty*, *50*(1), 1–34.

Danan, E., & Ziegelmeyer, A. (2006). *Are preferences complete? an experimental measurement of indecisiveness under risk* (resreport No. 2006-01). Max Planck Institue of Economics, Strategic Interaction Group. Retrieved from https://ideas.repec.org/p/esi/discus/2006-01.html

Dean, M., & Neligh, N. L. (in press). Experimental tests of rational inattention. *Journal of Political Economy*. doi: https://doi.org/10.1086/725174

Denti, T. (2022). Posterior separable cost of information. *American Economic Review*, *112*(10), 3215-3259.

Denti, T., & Pomatto, L. (2022). Model and predictive uncertainty: A foundation for smooth ambiguity preferences. *Econometrica*, *90*(2), 551–584.

Dewan, A., & Neligh, N. (2020). Estimating information cost functions in models of rational inattention. *Journal of Economic Theory*, *187*, 105011.

Dimmock, S. G., Kouwenberg, R., & Wakker, P. P. (2016). Ambiguity attitudes in a large representative sample. *Management Science*, *62*(5), 1363–1380.

Dwenger, N., Kübler, D., & Weizsäcker, G. (2018). Flipping a coin: Evidence from university applications. *Journal of Public Economics*, *167*, 240–250.

Eliaz, K., & Ok, E. A. (2006). Indifference or indecisiveness? choice-theoretic foundations of incomplete preferences. *Games and Economic Behavior*, *56*(1), 61–86.

Enke, B., & Graeber, T. (in press). Cognitive uncertainty. *Quarterly Journal of Economics*.

Faro, J. H. (2015). Variational bewley preferences. *Journal of Economic Theory*, *157*, 699–729.

Feldman, P., & Rehbeck, J. (2022). Revealing a preference for mixtures: An experimental study of risk. *Quantitative Economics*, *13*(2), 761–786.

Fudenberg, D., Iijima, R., & Strzalecki, T. (2015). Stochastic choice and revealed perturbed utility. *Econometrica*, *83*(6), 2371–2409.

Galaabaatar, T., & Karni, E. (2013). Subjective expected utility with incomplete preferences. *Econometrica*, *81*(1), 255–284.

Gerasimou, G. (2018). Indecisiveness, undesirability and overload revealed through rational choice deferral. *The Economic Journal*, *128*(614), 2450–2479.

Gerasimou, G. (2021). *Model-rich approach for eliciting possibly weak or incomplete preferences: evidence from a multi-valued choice experiment* (resreport). University of St. Andrews. doi: https://dx.doi.org/10.2139/ssrn.3972706

Ghirardato, P., Maccheroni, F., & Marinacci, M. (2004). Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory*, *118*(2), 133–173.

Gilboa, I., Maccheroni, F., Marinacci, M., & Schmeidler, D. (2010). Objective and subjective rationality in a multiple prior model. *Econometrica*, *78*(2), 755–770.

Gilboa, I., & Schmeidler, D. (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, *18*(2), 141–153.

Greiner, B. (2015). Subject pool recruitment procedures: organizing experiments with orsee. *Journal of the Economic Science Association*, *1*(1), 114–125.

Halevy, Y., Walker-Jones, D., & Zrill, L. (2023). Difficult decisions. *Unpublished Manuscript, July*.

Karni, E. (2009). A mechanism for eliciting probabilities. *Econometrica*, *77*(2), 603–606.

Karni, E. (2022). Incomplete risk attitudes and random choice behavior: an elicitation mechanism. *Theory and Decision*, *92*(3), 677–687.

Karni, E. (2023). *Irresolute choice behavior* (resreport). Johns Hopkins University. Retrieved from `https://www.econ2.jhu.edu/people/karni/irresolutechoice.pdf`

Khaw, M. W., Li, Z., & Woodford, M. (2021). Cognitive imprecision and small-stakes risk aversion. *The Review of Economic Studies*, *88*(4), 1979–2013.

Klibanoff, P., Marinacci, M., & Mukerji, S. (2005). A smooth model of decision making under ambiguity. *Econometrica*, *73*(6), 1849–1892.

Kreps, D. M. (1979). A representation theorem for "preference for flexibility". *Econometrica*, *47*(3), 565–577.

Maccheroni, F., Marinacci, M., & Rustichini, A. (2006). Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, *74*(6), 1447–1498.

Machina, M. J. (1985). Stochastic choice functions generated from deterministic preferences over lotteries. *The Economic Journal*, *95*(379), 575–594.

Matějka, F., & McKay, A. (2015). Rational inattention to discrete choices: A new foundation for the multinomial logit model. *American Economic Review*, *105*(1), 272–98.

Nau, R. (2006). The shape of incomplete preferences. *The Annals of Statistics*, *34*(5), 2430–2448.

Nielsen, K., & Rigotti, L. (2022). *Revealed incomplete preferences* (resreport). doi: https://doi.org/10.48550/arXiv.2205.08584

Ok, E. A., Ortoleva, P., & Riella, G. (2012). Incomplete preferences under uncertainty: Indecisiveness in beliefs versus tastes. *Econometrica*, *80*(4), 1791–1808.

Ok, E. A., & Tserenjigmid, G. (2022). Indifference, indecisiveness, experimentation, and stochastic choice. *Theoretical Economics*, *17*(2), 651–686.

Riella, G. (2015). On the representation of incomplete preferences under uncertainty with indecisiveness in tastes and beliefs. *Economic Theory*, *58*(3), 571–600.

Sopher, B., & Narramore, J. M. (2000). Stochastic choice and consistency in decision making under risk: An experimental study. *Theory and Decision*, *48*(4), 323–350.

Tversky, A. (1972). Elimination by aspects: a theory of choice. *Psychological Review*, *79*(4), 281–299.

von Neumann, J., & Morgenstern, O. (2007). *Theory of games and economic behavior.* Princeton University Press. (60th anniversary Edition with an introduction by Harold W. Kuhn and an afterword by Ariel Rubinstein)

Woodford, M. (2020). Modeling imprecision in perception, valuation, and choice. *Annual Review of Economics*, *12*, 579–601.

A More Theory Results (for online publication)

We begin with three simple lemmas that we use without citation throughout the paper due to their simple and intuitive nature. These lemmas show that the strict preference versions of TR and WCI are implied by TR and WCI respectively, and that the strict preference version of CON is implied by COM, CON, and TR.

Lemma 1. If the preferences of the DM satisfy TR, then for all $f,\, g,\, h \in \mathcal{F}$, if $f \succ g$ and $g \succeq h$, or $f \succeq g$ and $g \succ h$, then $f \succ h$.

Proof. First, assume $f \succ g$ and $g \succeq h$. TR tells us $f \succeq h$. If it is not the case that $f \succ h$ then $f \sim h$ and we can reach a contradiction since $g \succeq h$ and then TR tells us $g \succeq f$ which contradicts $f \succ g$.

Second, assume $f \succeq g$ and $g \succ h$. TR tells us $f \succeq h$. If it is not the case that $f \succ h$ then $f \sim h$ and we can reach a contradiction since $f \succeq g$ and then TR tells us $h \succeq g$, which contradicts $g \succ h$. ■

Lemma 2. If the preferences of the DM satisfy WCI, then for all $f,\, g, \in \mathcal{F}$, $x,\, y \in X$, and $\alpha \in (0,\, 1)$:

$$\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x \implies \alpha f + (1-\alpha)y \succ \alpha g + (1-\alpha)y.$$

Proof. Assume not, so $\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x$, and thus $\alpha g + (1-\alpha)y \sim \alpha f+(1-\alpha)y$ (because $\alpha f+(1-\alpha)x \succ \alpha g+(1-\alpha)x$, WCI tells us that $\alpha f+(1-\alpha)y \succeq \alpha g+(1-\alpha)y$, so, if it is not the case that $\alpha f+(1-\alpha)y \succ \alpha g+(1-\alpha)y$, then it must be that $\alpha g+(1-\alpha)y \sim \alpha f+(1-\alpha)y$), but then WCI tells us $\alpha g+(1-\alpha)x \succeq \alpha f+(1-\alpha)x$, and we immediately have a contradiction with $\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x$. ■

Lemma 3. If the preferences of the DM satisfy COM, CON, and TR, then for all $f,\, g,\, h \in \mathcal{F}$, if $f,\, h \succ g$, then for all $\alpha \in (0,\, 1)$: $\alpha f + (1-\alpha)h \succ g$.

Proof. Suppose $f,\, h \succ g$. COM tells us either $f \succeq h$ or $h \succeq f$. Without loss of generality assume $f \succeq h$. Then CON tell us that for all $\alpha \in (0,\, 1)$: $\alpha f + (1-\alpha)h \succeq h$ and thus TR (and Lemma 1) tells us for all $\alpha \in (0,\, 1)$: $\alpha f + (1-\alpha)h \succ g$. ■

Next we prove Proposition 1, which indicates the major relationships between the axioms, the VP model (Maccheroni et al., 2006), the UAP model Cerreia-Vioglio et al. (2011), the Smooth Ambiguity Preferences Model (Klibanoff et al., 2005; Denti & Pomatto, 2022), and SAAP (Klibanoff et al., 2005; Denti & Pomatto, 2022).

Proof of Proposition 1. Define X and S as in Section 2, and assume WP. COM and TR are together strictly weaker than axiom A.1 (Weak Order) in the work of Cerreia-Vioglio et al. (2011) and are together strictly weaker than Axiom 1 in the work of Denti and Pomatto (2022). WSTR is implied by axioms A.1 (Weak Order) and A.2 (Monotonicity) in the work of Cerreia-Vioglio et al. (2011). CON is implied by A.1 (Weak Order), A.3 (Convexity), and A.5 (Continuity), in the work of Cerreia-Vioglio et al. (2011), as is shown by Lemma 56 in their work.

Axioms A.1 through A.6 in the work of Maccheroni et al. (2006) imply axioms A.1 through A.5 in the work of Cerreia-Vioglio et al. (2011), this implication is trivially true for each of the mentioned axioms from the work of Cerreia-Vioglio et al. (2011) except for A.4 (Risk Independence), whose implication is trivial given WP and our particular X. Further, WCI is axiom A.2 (Weak Certainty Independence) in the work of Maccheroni et al. (2006). The implication of STR via axioms A.1 through A.6 in the work of Maccheroni et al. (2006) is trivial given Theorem 3 from their work and WP.

STR is implied by Axiom 1 and Axiom 2 in the work of Denti and Pomatto (2022). Given Definition 1 in the work of Denti and Pomatto (2022), which provides a representation of Smooth Ambiguity Preferences, it is trivial to show that SAAP, which uses this representation and further imposes that the function ϕ is concave, satisfies CON. ■

The rest of this appendix presents results in the continuous slider setting introduced in Section 2 and Section 3.

Continuous Interval Results

Next, we introduce a proposition that relates WP to the work of Cerreia-Vioglio et al. (2011). The proposition demonstrates that WP is a quite weak assumption about preferences.

Proposition 2. *Given X and S as defined above, if always winning is preferred to always losing, namely $y = 1 \succeq x = 0$, then WP is implied by axioms A.1 through A.5 from the work of Cerreia-Vioglio et al. (2011).*

Proof. Define X and S as in Section 2. Assume $y = 1 \succeq x = 0$ and that axioms A.1 through A.5 from the work of Cerreia-Vioglio et al. (2011) are satisfied. It must then be that $y = 1 \succ x = 0$, because if $y \sim x$ then Axiom A.1 (Weak Order, transitivity part) and Axiom A.4 (Risk Independence) can be used to show that for all $z \in [0, 1]$ we have $z \sim x \sim y$, because Axiom A.4 (Risk Independence) tells us $0 \sim 1 \Rightarrow 0 \sim \alpha$ for all $\alpha \in (0, 1)$. But, $z \sim x \sim y$ is not possible since Axiom A.1 (Weak Order, transitivity part) and Axiom A.2 (Monotonicity) would then tell us the DM is indifferent between all acts and that contradicts Axiom A.1 (Weak Order, nontrivial part), so $y = 1 \succ x = 0$.

Now assume that $\exists z, w \in X$ such that $z < w$ and $z \succeq w$, and we shall reach a contradiction. By Axiom A.1 (Weak Order, completeness part) either $x \succeq z$ or $z \succeq x$ and $x \succeq w$ or $w \succeq x$. If $x \succ z$ then there is $v > z$ such that $v \sim x$ by Axiom A.5 (Continuity) since $y \succ x \succ z$. If, instead, $z \succeq x$ and $x \succeq w$ then there is $v \geq z$ such that $v \sim x$ by Axiom A.5 (Continuity). If, instead, $z \succeq x$ and $w \succeq x$ then there is $v \leq z$ such that $v \sim w$ by Axiom A.5 (Continuity). In any of these cases we have $v, u \in X$ such that $v \neq u$ and yet $v \sim u$. This is problematic since then Axiom A.1 (Weak Order, transitivity part) and Axiom A.4 (Risk Independence) can be used to show if $t, s \in (0, 1)$ then $t \sim s$, since for $t \in (v, u)$ we have $t \sim v$ since $\alpha u + (1-\alpha)v \sim \alpha v + (1-\alpha)v = v$ for all $\alpha \in (0, 1)$, and then the indifference region $[v, u]$ (supposing without loss that $u > v$) can be iteratively expanded towards $(0, 1)$ since $\alpha v + (1-\alpha)0 \sim \alpha u + (1-\alpha)0$ and $\alpha v + (1-\alpha)1 \sim \alpha u + (1-\alpha)1$ for all $\alpha \in (0, 1)$. However, $t \sim s$ for all $t, s \in (0, 1)$ combined with the fact that $y = 1 \succ x = 0$ creates a contradiction with Axiom A.1 (Weak order, transitivity and completeness parts) and Axiom A.5 (Continuity). □

Next, we provide the proofs for Theorem 3 and Theorem 4.

Proof of Theorem 3. If for all $x \in (\frac{1}{2}, 1]$ the DM's preferences are such that $\frac{1}{4}x \succ \frac{1}{4}R$ and $\frac{1}{4}x \succ \frac{1}{4}L$, then COM, CON, and WCI, tell us $\frac{1}{2}x \succ \frac{1}{2}R$ and $\frac{1}{2}x \succ \frac{1}{2}L$ for all $x \in (\frac{1}{2}, 1]$, because if not $\frac{1}{2}R \succeq \frac{1}{2}x \Rightarrow \frac{1}{4}R + \frac{1}{4}x \succeq \frac{1}{2}x \Rightarrow \frac{1}{4}R \succeq \frac{1}{4}x$ and $\frac{1}{2}L \succeq \frac{1}{2}x \Rightarrow \frac{1}{4}L + \frac{1}{4}x \succeq \frac{1}{2}x \Rightarrow \frac{1}{4}L \succeq \frac{1}{4}x$, so we would already have a contradiction, thus, COM and CON tell us $\frac{1}{4}R + \frac{1}{4}x \succeq \frac{1}{2}R$ and $\frac{1}{4}L + \frac{1}{4}x \succeq \frac{1}{2}L$, and thus, since this is true for all such x, STR tells us $\frac{1}{4}R + \frac{1}{4}x \succ \frac{1}{2}R$ and $\frac{1}{4}L + \frac{1}{4}x \succ \frac{1}{2}L$ for all $x \in (\frac{1}{2}, 1]$ since for all $\tilde{x} \in (\frac{1}{2}, x)$ $\frac{1}{4}R + \frac{1}{4}\tilde{x} \succeq \frac{1}{2}R$ and $\frac{1}{4}L + \frac{1}{4}\tilde{x} \succeq \frac{1}{2}L$.

So, if $\frac{1}{4}x \succ \frac{1}{4}R$ and $\frac{1}{4}x \succ \frac{1}{4}L$ for all $x \in (\frac{1}{2}, 1]$ (assumed throughout this paragraph) then $l_L = u_L = l_R = u_R = \frac{1}{2}$. This is true because, to begin with, if $\min(l_L, l_R) < \frac{1}{2}$ there exists small $\epsilon > 0$ such that $\min(l_L, l_R) + \epsilon < \frac{1}{2}$ and for all $y \in [\min(l_L, l_R), \min(l_L, l_R) + \epsilon]$ COM, STR, and WCI, tell us:

$$\frac{1}{2} \succ \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y \text{ because } \frac{1}{4}(1-y) \succ \frac{1}{4}R,$$

$$\frac{1}{2} \succ \frac{1}{4}L + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y \text{ because } \frac{1}{4}(1-y) \succ \frac{1}{4}L,$$

$$\text{and } \frac{1}{2} \succ \frac{1}{2}\frac{1}{2} + \frac{1}{2}y,$$

so if the minimum lower bound is strictly less than the minimum upper bound the DM could strictly benefit from increasing their minimum lower bound (if $l_R > l_L$ then increasing l_L a small amount changes the chosen act locally from $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ to $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$, while if $l_R < l_L$ then increasing l_R a small amount similarly changes the chosen act locally from $\frac{1}{4}L + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ to $\frac{1}{2}$), or by increasing both if they are equal (if $l_R = l_L$ then increasing both l_R and l_L the same small amount changes the chosen act locally from $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$ to $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$), and COM, STR, and WCI, tell us (for $y < \frac{1}{2}$):

$$\frac{1}{2}\frac{1}{2} + \frac{1}{2}y \succ y,$$

$$\frac{1}{2}\frac{1}{2} + \frac{1}{2}y \succ \frac{1}{4}R + \frac{3}{4}y \text{ again because } \frac{1}{4}(1-y) \succ \frac{1}{4}R,$$

$$\frac{1}{2}\frac{1}{2} + \frac{1}{2}y \succ \frac{1}{4}L + \frac{3}{4}y \text{ again because } \frac{1}{4}(1-y) \succ \frac{1}{4}L,$$

$$\frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y \succ \frac{1}{2}R + \frac{1}{2}y \text{ because } \frac{1}{4}R + \frac{1}{4}(1-y) \succ \frac{1}{2}R,$$

$$\text{and } \frac{1}{4}L + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y \succ \frac{1}{2}L + \frac{1}{2}y \text{ because } \frac{1}{4}L + \frac{1}{4}(1-y) \succ \frac{1}{2}L,$$

so the DM could strictly benefit from increasing their minimum upper bound(s) if one or more of them is equal to $\min(l_L, l_R)$ (if $u_L = u_R = \min(l_L, l_R) < \frac{1}{2}$ then increasing u_L and u_R the same small amount changes the chosen act locally from y to $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$, if $u_L = \min(l_L, l_R) < u_R$ then increasing u_L a small amount changes the chosen act locally, depending on if $u_L \geq l_R$ or $u_L < l_R$, from $\frac{1}{4}R + \frac{3}{4}y$ to $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$ or from $\frac{1}{2}R + \frac{1}{2}y$ to $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y =$

$\frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$, and if $u_R = \min(l_L, l_R) < u_L$ then increasing u_R a small amount changes the chosen act locally, depending on if $u_R \geq l_L$ or $u_R < l_L$, from $\frac{1}{4}L + \frac{3}{4}y$ to $\frac{1}{2}(\frac{1}{2}R+\frac{1}{2}L)+\frac{1}{2}y = \frac{1}{2}\frac{1}{2}+\frac{1}{2}y$ or from $\frac{1}{2}L+\frac{1}{2}y$ to $\frac{1}{4}L+\frac{1}{2}(\frac{1}{2}R+\frac{1}{2}L)+\frac{1}{4}y = \frac{1}{4}L+\frac{1}{2}\frac{1}{2}+\frac{1}{4}y)$. Further, if $\max(l_L, l_R) > \frac{1}{2}$, since there exists small $\epsilon > 0$ such that $\max(l_L, l_R)-\epsilon > \frac{1}{2}$ and for all $y \in [\max(l_L, l_R) - \epsilon, \max(l_L, l_R)]$ COM, STR, and WCI, tell us (given what we showed in the previous paragraph):

$$\frac{1}{2}y + \frac{1}{2}\frac{1}{2} \succ \frac{1}{2},$$

$$\frac{1}{2}y + \frac{1}{2}\frac{1}{2} \succ \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y \text{ because now (for these } y) \ \frac{1}{4}y \succ \frac{1}{4}R,$$

$$\frac{1}{2}y + \frac{1}{2}\frac{1}{2} \succ \frac{1}{4}L + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y \text{ because now (for these } y) \ \frac{1}{4}y \succ \frac{1}{4}L,$$

$$\frac{1}{4}R + \frac{3}{4}y \succ \frac{1}{2}R + \frac{1}{2}y \text{ because now (for these } y) \ \frac{1}{4}R + \frac{1}{4}y \succ \frac{1}{2}R$$

$$\text{and } \frac{1}{4}L + \frac{3}{4}y \succ \frac{1}{2}L + \frac{1}{2}y \text{ because now (for these } y) \ \frac{1}{4}L + \frac{1}{4}y \succ \frac{1}{2}L,$$

so the DM could do strictly better by lowering $\max(l_L, l_R)$ (if $l_R > l_L$ then decreasing l_R a small amount changes the chosen act locally, depending on if $l_R > u_L$ or $l_R \leq u_L$, from $\frac{1}{2}R + \frac{1}{2}y$ to $\frac{1}{4}R + \frac{3}{4}y$ or from $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ to $\frac{1}{2}y+\frac{1}{2}(\frac{1}{2}R+\frac{1}{2}L) = \frac{1}{2}y+\frac{1}{2}\frac{1}{2}$, and if $l_R < l_L$ then decreasing l_L a small amount changes the chosen act locally, depending on if $l_L > u_R$ or $l_L \leq u_R$, from $\frac{1}{2}L + \frac{1}{2}y$ to $\frac{1}{4}L + \frac{3}{4}y$ or from $\frac{1}{4}L + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}L + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ to $\frac{1}{2}y + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) = \frac{1}{2}y + \frac{1}{2}\frac{1}{2})$, or both if they are equal (if $l_R = l_L$ then decreasing both l_R and l_L the same small amount changes the chosen act locally from $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$ to $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y)$, so we can conclude that $\max(l_L, l_R) \leq \frac{1}{2}$, and if $\max(u_L, u_R) > \frac{1}{2}$ then we know $\max(u_L, u_R) > \max(l_L, l_R)$ and since there exists small $\epsilon > 0$ such that $\max(u_L, u_R) - \epsilon > \frac{1}{2}$ and for all $z \in [\max(u_L, u_R) - \epsilon, \max(u_L, u_R)]$ COM, STR, and WCI, tell us:

$$z \succ \frac{1}{2}\frac{1}{2} + \frac{1}{2}z, \text{ because } z > \frac{1}{2}$$

$$z \succ \frac{1}{4}R + \frac{3}{4}z, \text{ because (for these } z) \ \frac{1}{4}z \succ \frac{1}{4}R$$

$$\text{and } z \succ \frac{1}{4}L + \frac{3}{4}z \text{ because (for these } z) \ \frac{1}{4}z \succ \frac{1}{4}L,$$

so the DM could do strictly better by lowering $\max(u_L, u_R)$ (if $u_R > u_L$ then decreasing u_R a small amount changes the chosen act locally from $\frac{1}{4}R + \frac{3}{4}z$ to z and if $u_R < u_L$ then decreasing u_L a small amount changes the chosen act locally from $\frac{1}{4}L + \frac{3}{4}z$ to z) or both if they are equal (if $u_R = u_L$ then decreasing both u_R and u_L the same small amount changes the chosen act locally from $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}z = \frac{1}{2}\frac{1}{2} + \frac{1}{2}z$ to z).

Assume for the rest of the proof $\exists x \in (\frac{1}{2}, 1]$ and $f \in \{L, R\}$ such that $\frac{1}{4}f \succeq \frac{1}{4}x$, then for $e \in \{L, R\} \backslash \{f\}$ we know $\frac{1}{4}y \succ \frac{1}{4}e$ for all $y \in (\frac{1}{2}, 1]$ because if not COM, STR, and CON, tell us $\frac{1}{8}f + \frac{1}{8}e = \frac{1}{4}\frac{1}{2} \succeq \frac{1}{4}\min(x, y)$, which violates COM and STR. Assume without loss that $\frac{1}{4}R \succeq \frac{1}{4}x$ for some $x \in (\frac{1}{2}, 1]$, thus STR tells us $\frac{1}{4}R \succ \frac{1}{4}x$ for some $x \in (\frac{1}{2}, 1]$ and $\frac{1}{4}y \succ \frac{1}{4}L$ for all $y \in (\frac{1}{2}, 1]$, and thus COM, CON, and WCI, tell us that for all such y that $\frac{1}{4}L + \frac{1}{4}y \succeq \frac{1}{2}L$ (because we cannot have that $\frac{1}{2}L \succ \frac{1}{4}L + \frac{1}{4}y$ and $\frac{1}{2}y \succ \frac{1}{4}L + \frac{1}{4}y$), and since this is true for all such y STR tells us that for all such y that $\frac{1}{4}L + \frac{1}{4}y \succ \frac{1}{2}L$, and thus COM, CON, and WCI tell us for all such y that $\frac{1}{2}y \succeq \frac{1}{2}L$ (because, otherwise, COM implies $\frac{1}{2}L \succ \frac{1}{2}y$, and then COM and CON imply $\frac{1}{4}L + \frac{1}{4}y \succeq \frac{1}{2}y$, and then WCI implies $\frac{1}{4}L \succeq \frac{1}{4}y$, which contradicts $\frac{1}{4}y \succ \frac{1}{4}L$) and since this is true for all such y STR tells us that for all such y that $\frac{1}{2}y \succ \frac{1}{2}L$.

It must then be that $u_R \geq u_L$ because otherwise (if $u_R < u_L$) COM, STR, and WCI, tell us, since for all $y \in (\frac{1}{2}, 1]$:

$$y \succ \frac{1}{4}L + \frac{3}{4}y \text{ because (for these } y\text{, based on previous paragraph) } \frac{1}{4}y \succ \frac{1}{4}L$$

$$\frac{1}{4}L + \frac{3}{4}y \succ \frac{1}{2}L + \frac{1}{2}y \text{ because (for these } y\text{, based on previous paragraph) } \frac{1}{4}L + \frac{1}{4}y \succ \frac{1}{2}L,$$

$$\text{and } y \succ \frac{1}{2}L + \frac{1}{2}y \text{ because (for these } y\text{, based on previous paragraph) } \frac{1}{2}y \succ \frac{1}{2}L,$$

if $u_L > \frac{1}{2}$ the DM could do strictly better by decreasing u_L if $u_L > l_L$ (if $u_L > l_L$ then decreasing u_L a small amount changes the chosen act locally from $\frac{1}{4}L + \frac{3}{4}y$ to y) and by decreasing both u_L and l_L if $l_L = u_L$ (if $l_L = u_L$ then decreasing both u_L and l_L the same small amount changes the chosen act locally, depending on if the location is equal to the new u_L or the old u_L or between the two, from $\frac{1}{2}L + \frac{1}{2}y$ to $\frac{1}{4}L + \frac{3}{4}y$ or from $\frac{1}{4}L + \frac{3}{4}y$ to y or from $\frac{1}{2}L + \frac{1}{2}y$ to y) while, for similar reasons, if $u_L \leq \frac{1}{2}$ the DM could do strictly better by increasing u_R (if $u_R < u_L \leq \frac{1}{2}$ then increasing u_R a small amount changes the chosen act locally, depending on if $u_R < l_L$ or $u_R \geq l_L$, from $\frac{1}{2}L + \frac{1}{2}y$ to $\frac{1}{4}L + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}L + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ for $y < \frac{1}{2}$ or from $\frac{1}{4}L + \frac{3}{4}y$

to $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$ for $y < \frac{1}{2}$).

It must also then be that $l_R \geq l_L$ because, again, COM, STR, and WCI tell us that, since:

$$\frac{1}{2} \succ \frac{1}{4}L + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y \text{ for } y < \frac{1}{2} \text{ because we know } \frac{1}{4}(1-y) \succ \frac{1}{4}L,$$

$$\text{and } \frac{1}{2}\frac{1}{2} + \frac{1}{2}y \succ \frac{1}{4}L + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y \text{ for } y > \frac{1}{2} \text{ because we know } \frac{1}{4}y \succ \frac{1}{4}L,$$

if $l_R < l_L$ (so $l_R < u_R$ since we showed $u_R \geq u_L$) and $l_L \leq \frac{1}{2}$ then the DM could do strictly better by increasing l_R (locally this changes the chosen act from $\frac{1}{4}L + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}L + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ to $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$), while if $l_R < l_L$ (so $l_R < u_R$ since we showed $u_R \geq u_L$) and $l_L > \frac{1}{2}$ then the DM could do strictly better by decreasing l_L (locally this changes the chosen act from $\frac{1}{4}L + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}L + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ to $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$). It must also then be that $l_R \geq u_L$, because COM, STR, and WCI, tell us:

$$\frac{1}{2} \succ \frac{1}{4}L + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y \text{ for } y < \frac{1}{2} \text{ because we know } \frac{1}{4}(1-y) \succ \frac{1}{4}L,$$

$$\frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y \succ \frac{1}{2}\frac{1}{2} + \frac{1}{2}y \text{ for } y < \frac{1}{2} \text{ because we know } \frac{1}{4}R \succ \frac{1}{4}y,$$

$$\text{and } \frac{1}{4}R + \frac{3}{4}y \succ \frac{1}{2}\frac{1}{2} + \frac{1}{2}y \text{ for } y > \frac{1}{2} \text{ because we know } \frac{1}{4}R \succ \frac{1}{4}(1-y),$$

so the DM could otherwise (if $l_R < u_L \leq u_R$) do strictly better from at least one of increasing l_R (if $u_L \leq \frac{1}{2}$ then increasing l_R locally changes the chosen act, since $l_R \geq l_L$, from $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$ to $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$) or decreasing u_L (if $u_L > \frac{1}{2}$ then decreasing u_L locally changes the chosen act from $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$ to $\frac{1}{4}R + \frac{3}{4}y$).

So, $l_R \geq u_L$ (notice that this establishes that we cannot have an interval that is contained in the interior of the other interval). If $l_R > u_L$ then the fact that the DM does not decrease l_R (which would locally change the chosen act from $\frac{1}{2}R + \frac{1}{2}z$ to $\frac{1}{4}R + \frac{3}{4}z$), COM, STR, and WCI, tell us for all $\epsilon > 0$ there is $z \in (l_R - \epsilon, l_R)$ such that (using WCI and then COM and CON):

$$\frac{1}{2}R + \frac{1}{2}z \succ \frac{1}{4}R + \frac{3}{4}z$$

$$\Rightarrow \frac{1}{2}R \succ \frac{1}{4}R + \frac{1}{4}z \Rightarrow \frac{1}{2}R \succ \frac{1}{2}z \text{ and } \frac{1}{4}R + \frac{1}{4}z \succeq \frac{1}{2}z,$$

and the fact that the DM does not increase u_L (which would locally change the chosen act from $\frac{1}{2}R + \frac{1}{2}y$ to $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$), COM, STR, and WCI, tell us for all $\epsilon > 0$ there is $q \in (u_L, u_L + \epsilon)$ such that (using WCI then COM and CON):

$$\frac{1}{2}R + \frac{1}{2}q \succ \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}q$$

$$\Rightarrow \frac{1}{2}R \succ \frac{1}{4}R + \frac{1}{4}(1-q) \Rightarrow \frac{1}{2}R \succ \frac{1}{2}(1-q) \text{ and } \frac{1}{4}R + \frac{1}{4}(1-q) \succeq \frac{1}{2}(1-q),$$

so if $l_R > 1 - u_L$, which thus implies $l_R > \frac{1}{2}$, then STR and WCI tell us the DM could do strictly better by decreasing u_L if $u_L > l_L$ (which locally changes the chosen act from $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ to $\frac{1}{2}R + \frac{1}{2}y$) and by decreasing u_L and l_L if $l_L = u_L$ (which locally changes the chosen act, depending on if the location is equal to the new u_L the old u_L or between the two, from $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$ to $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ or from $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ to $\frac{1}{2}R + \frac{1}{2}y$ or from $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$ to $\frac{1}{2}R + \frac{1}{2}y$), while if $l_R < 1 - u_L$, which thus implies $u_L < \frac{1}{2}$, then STR and WCI tell us the DM could do strictly better by increasing l_R if $l_R < u_R$ (which locally changes the chosen act from $\frac{1}{4}R + \frac{3}{4}y$ to $\frac{1}{2}R + \frac{1}{2}y$) and by increasing l_R and u_R if $l_R = u_R$ (which locally changes the chosen act, depending on if the location is that of the new u_R or the old u_R or between the two, from y to $\frac{1}{4}R + \frac{3}{4}y$ or from $\frac{1}{4}R + \frac{3}{4}y$ to $\frac{1}{2}R + \frac{1}{2}y$ or from y to $\frac{1}{2}R + \frac{1}{2}y$). So, if $l_R > u_L$ then $l_R = 1 - u_L$.

If $u_R = 1$ then COM, STR, CON, and WCI, tell us it cannot be that $l_L = l_R = u_R$ as then the DM could do strictly better by reducing l_L and l_R (locally this changes the chosen act from $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$ to $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$), and thus if $u_R = 1$, for all small $\epsilon > 0$ there is $x \in (1 - \epsilon, 1)$ such that:

$$\frac{1}{4}R \succ \frac{1}{4}x,$$

since otherwise there is small $\delta > 0$ such that for all $x \in (1 - \delta, 1)$:

$$\frac{1}{4}x \succ \frac{1}{4}R, \ \frac{1}{2}x \succ \frac{1}{2}R \text{ and } \frac{1}{4}R + \frac{1}{4}x \succ \frac{1}{2}R,$$

and the DM could do strictly better by decreasing u_R if $u_R > l_R$ (locally this changes the chosen act from $\frac{1}{4}R + \frac{3}{4}y$ to y), by decreasing l_R if $u_R = l_R > u_L$ (locally this changes the chosen act from $\frac{1}{2}R + \frac{1}{2}y$ to $\frac{1}{4}R + \frac{3}{4}y$), and by decreasing l_R if $l_R = u_R = u_L > l_L$ (locally this changes the chosen act from $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ to $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$), and thus $l_L = 0$ because otherwise the DM could do strictly better by reducing l_L (locally this changes the chosen act from $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$ to $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$). If $l_L = 0$ then COM, STR, CON, and WCI, tell us it cannot be that $u_R = u_L = l_L$ since then the DM could do strictly better by increasing u_L and u_R (locally this changes the chosen act from y to $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$), and thus for all small $\epsilon > 0$ there is $y \in (0, \epsilon)$ such that:

$$\frac{1}{4}R + \frac{1}{4}y \succ \frac{1}{2}\frac{1}{2} \Rightarrow \frac{1}{4}R \succ \frac{1}{4}(1-y),$$

since otherwise there is small $\delta > 0$ such that for all $y \in (0, \delta)$:

$$\frac{1}{2}\frac{1}{2} \succ \frac{1}{4}R + \frac{1}{4}y,\ \frac{1}{2} \succ \frac{1}{2}R + \frac{1}{2}y \text{ and } \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y \succ \frac{1}{2}R + \frac{1}{2}y,$$

and the DM could do strictly better by increasing l_L if $l_L < u_L$ (locally this changes the chosen act from $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ to $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$), by increasing u_L if $l_L = u_L < l_R$ (locally this changes the chosen act from $\frac{1}{2}R + \frac{1}{2}y$ to $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$), and by increasing u_L if $u_L = l_R = l_L < u_R$ (locally this changes the chosen act from $\frac{1}{4}R + \frac{3}{4}y$ to $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$), and thus $u_R = 1$ because otherwise the DM could do strictly better by increasing u_R (locally this changes the chosen act from y to $\frac{1}{4}R + \frac{3}{4}y$). If $u_R < 1$ then, as is implied by our work above, $l_L > 0$, and the fact that the DM does not increase u_R (which locally changes the chosen act from y to $\frac{1}{4}R + \frac{3}{4}y$) or decrease l_L (which locally changes the chosen act from $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$ to $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$), COM, STR, and WCI, tell us for all small $\epsilon > 0$ there is $y \in [u_R, u_R + \epsilon]$ such that:

$$y \succ \frac{1}{4}R + \frac{3}{4}y,$$

and there is $z \in [l_L - \epsilon, l_L]$ such that:

$$\frac{1}{2} \succ \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}z.$$

Thus, COM, STR, CON, and WCI, tell us that if $u_R < 1 - l_L$ (which, given what we have shown, implies $l_L < \frac{1}{2}$) the DM could do strictly better by increasing l_L if $u_L > l_L$ (which locally changes the chosen act from $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ to $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$), or by increasing u_L if $l_L = u_L < u_R$ (which locally changes the chosen act from $\frac{1}{4}R + \frac{3}{4}y$ to $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$ since $u_L = l_L < 1 - u_R \Rightarrow l_R = u_L$ given that we showed above that if $l_R > u_L$ then $l_R + u_L = 1 \Rightarrow u_R + u_L \geq 1$), or by increasing u_R and u_L if $l_L = u_L = u_R$ (which locally changes the chosen act from y to $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$). If, instead, $u_R > 1 - l_L$ (which, given what we have shown, implies $u_R > \frac{1}{2}$) COM, STR, CON, and WCI, tell us that the DM could do strictly better by decreasing u_R if $u_R > l_R$ (which locally changes the chosen act from $\frac{1}{4}R + \frac{3}{4}y$ to y), or by decreasing l_R if $u_R = l_R > l_L$ (which locally changes the chosen act from $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ to $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$ since $u_R = l_R > 1 - l_L \Rightarrow l_R = u_L$ given what we showed above that if $l_R > u_L$ then $l_R + u_L = 1 \Rightarrow l_R + l_L \leq 1$), or by decreasing l_R and l_L if $u_R = l_R = l_L$ (which locally changes the chosen lottery from $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$ to $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}y = \frac{1}{2}\frac{1}{2} + \frac{1}{2}y$). So, $u_R = 1 - l_L$. ■

Proof of Theorem 4. Notice that the implications of Theorem 3 hold in this setting since CI implies WCI when COM holds. Suppose the DM reports a non-degenerate interval and thus, given the result in Theorem 3, we must either have a unique maximum bound (and) or a unique minimum bound. Further, notice that the DM would never report one interval that is contained in the interior of their other interval, as is shown in the poof of Theorem 3. We can thus assume without loss of generality that $u_R > l_L$. If u_R is the unique maximal bound, by which we mean $u_R > \max(u_L, l_R)$, then the fact that the DM does not decrease u_R, COM, STR, and WCI, tell us that for all small $\epsilon > 0$ there is $y \in [u_R - \epsilon, u_R]$ such that:

$$\frac{1}{4}R + \frac{3}{4}y \succ y \Rightarrow \frac{1}{4}R \succ \frac{1}{4}y,$$

and then STR tells us there is a $z \in [u_R - \epsilon, 1]$ such that if $x < z$ then $\frac{1}{4}R \succ \frac{1}{4}x$ and if $x > z$ then $\frac{1}{4}x \succ \frac{1}{4}R$ (making sure in both cases that $\frac{1}{4}x \in X$), and the fact that the DM does not increase l_R (which, since Theorem 3 implies it must be that $u_L \leq l_R$, locally changes the chosen act strictly above u_L and weakly above l_R from $\frac{1}{4}R + \frac{3}{4}y$ to $\frac{1}{2}R + \frac{1}{2}y$ and potentially changes the chosen act at u_L, if $l_R = u_L$, from $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}l_R = \frac{1}{2}\frac{1}{2} + \frac{1}{2}l_R$ to $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}l_R = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}l_R$), COM, STR, and WCI, tell us that for all $\epsilon > 0$ there is $w \in [l_R, l_R + \epsilon]$ such that:

$$\frac{1}{4}R + \frac{3}{4}w \succ \frac{1}{2}R + \frac{1}{2}w,$$

because if for all $w \in [l_R, l_R + \epsilon]$:

$$\frac{1}{2}R + \frac{1}{2}w \succeq \frac{1}{4}R + \frac{3}{4}w$$

then for all $w \in [l_R, l_R + \epsilon)$:

$$\frac{1}{2}R + \frac{1}{2}w \succ \frac{1}{4}R + \frac{3}{4}w \Rightarrow \frac{1}{4}R + \frac{1}{4}w \succeq \frac{1}{2}w,$$

and then the DM could strictly benefit from increasing l_R, so thus TR and CI tell us that for all small $\epsilon > 0$ that:

$$\frac{1}{4}(z + \epsilon) + \frac{3}{4}w \succ \frac{1}{2}(z - \epsilon) + \frac{1}{2}w,$$

and taking ϵ to zero we get a contradiction with COM and STR. So, we must have $l_L < u_L \leq l_R = u_R$ (because we assumed there was a non-degenerate interval) and further Theorem 3 then requires $u_L = l_R$, but then l_L is the unique minimum bound and the fact that the DM does not increase l_L, COM, STR, and WCI, tell us that for all small $\epsilon > 0$ there is $q \in [l_L, l_L + \epsilon]$ such that:

$$\frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}q \succ \frac{1}{2} \Rightarrow \frac{1}{4}R \succ \frac{1}{4}(1 - q),$$

and then STR tell us there is a $k \in [0, l_L + \epsilon]$ such that if $x > k$ then $\frac{1}{4}R \succ \frac{1}{4}(1 - x)$ and if $x < k$ then $\frac{1}{4}(1 - x) \succ \frac{1}{4}R$ (making sure in both cases that $\frac{1}{4}(1 - x) \in X$), and the fact that the DM does not decrease u_L (which, since $u_L = l_R$, locally changes the chosen act below u_L from $\frac{1}{4}R + \frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{4}y = \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}y$ to $\frac{1}{2}R + \frac{1}{2}y$ and changes the chosen act at u_L from $\frac{1}{2}(\frac{1}{2}R + \frac{1}{2}L) + \frac{1}{2}u_L = \frac{1}{2}\frac{1}{2} + \frac{1}{2}u_L$ to $\frac{1}{4}R + \frac{3}{4}u_L$), COM, STR, and WCI, tell us that for all $\epsilon > 0$ there is $n \in [u_L - \epsilon, u_L]$ such that:

$$\frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}n \succ \frac{1}{2}R + \frac{1}{2}n,$$

because if for all $n \in [u_L - \epsilon,\, u_L]$:

$$\frac{1}{2}R + \frac{1}{2}n \succeq \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}n,$$

then for all $n \in (u_L - \epsilon,\, u_L]$:

$$\frac{1}{2}R + \frac{1}{2}n \succ \frac{1}{4}R + \frac{1}{2}\frac{1}{2} + \frac{1}{4}n \Rightarrow \frac{1}{4}R + \frac{1}{4}(1-n) \succeq \frac{1}{2}(1-n) \Rightarrow \frac{1}{4}R + \frac{3}{4}n \succeq \frac{1}{2}\frac{1}{2} + \frac{1}{2}n,$$

and then the DM could strictly benefit from decreasing u_L, so thus TR and CI tell us that for all small $\epsilon > 0$ that:

$$\frac{1}{4}(1-k+\epsilon) + \frac{1}{2}\frac{1}{2} + \frac{1}{4}n \succ \frac{1}{2}(1-k-\epsilon) + \frac{1}{2}n \Rightarrow \frac{1}{4}(1-k+\epsilon) + \frac{1}{4}(1-n) \succ \frac{1}{2}(1-k-\epsilon),$$

and taking ϵ to zero we get a contradiction with COM and STR. Thus, $u_R = l_R$ and $u_L = l_L$, and Theorem 3 tells us $\max(u_R,\, u_L) + \min(l_R,\, l_L) = 1$, so $u_R + u_L = 1$. ■

We are now ready to introduce the rest our theorems that imposes restrictions onto DM behavior when their preferences satisfy different subsets of our axioms.

Theorem 5. *If the DM assigns weight α to $f \in \{L,\, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy COM and STR, and the DM isolates the three choice problems, then if $\alpha = 1$ it must be that $l_f \geq \min(u_f,\, \frac{1}{2})$, while if $\alpha = \frac{3}{4}$ it must be that $l_f \leq \frac{1}{2}$.*

Proof. Assume without loss that $f = R$. If $\alpha = 1$ then COM tells us $R \succeq \frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{1}{2}$, so STR tells us for $x < \frac{1}{2}$ that $R \succ \frac{1}{2}R + \frac{1}{2}x$, so if $l_R < \min(u_R,\, \frac{1}{2})$ the DM could do strictly better by increasing l_R to $\min(u_R,\, \frac{1}{2})$. If $\alpha = \frac{3}{4}$ then COM tells us $\frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{1}{2} \succeq R$, so STR tells us for $x > \frac{1}{2}$ that $\frac{1}{2}R + \frac{1}{2}x \succ R$, so if $l_R > \frac{1}{2}$ the DM could do strictly better by decreasing l_R to $\frac{1}{2}$. □

Theorem 6. *If the DM selects $f \in \{L,\, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy WP, COM, WSTR, CON, and TR, and the DM isolates the three choice problems, then $u_f \geq \frac{1}{2}$.*

Proof. Without loss of generality assume $f = R$. Assume $u_R < \frac{1}{2}$ and we will reach a contradiction. The DM's selection of R tells us $R \succeq \frac{1}{4}L + \frac{3}{4}R = \frac{1}{2}\frac{1}{2} + \frac{1}{2}R$ by COM. So, $R \succeq \frac{1}{2}x + \frac{1}{2}R$, for $x < \frac{1}{2}$ by WSTR. The DM's selection of R also tells us

$R \succeq \frac{1}{2}L + \frac{1}{2}R = \frac{1}{2}$ by COM, thus $\frac{1}{2}\frac{1}{2} + \frac{1}{2}R \succeq \frac{1}{2}$ by COM and CON, and $R \succ x$ for $x < \frac{1}{2}$ by WP and TR. But, then we have the desired contradiction since the DM can do strictly better by increasing both u_R and l_R to $\frac{1}{2}$. $\square$

Theorem 7. *If the DM selects $f \in \{L, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy COM, STR, and CON, and the DM isolates the three choice problems, then the relevant lower bound l_f is such that $l_f \geq \frac{1}{2}$.*

Proof. Without loss of generality assume $f = R$. Assume $l_R < \frac{1}{2}$ and we will reach a contradiction. The DM's selection of R tells us $R \succeq \frac{1}{2}L + \frac{1}{2}R = \frac{1}{2}$ by COM, thus $\frac{1}{2}\frac{1}{2} + \frac{1}{2}R \succeq \frac{1}{2}$ by COM and CON, $R \succ x$, for $x < \frac{1}{2}$ by STR, and the DM's selection of R also tells us $R \succeq \frac{1}{4}L + \frac{3}{4}R = \frac{1}{2}\frac{1}{2} + \frac{1}{2}R$ by COM, so, $R \succ \frac{1}{2}x + \frac{1}{2}R$, for $x < \frac{1}{2}$ by STR. But, then the DM can do strictly better by increasing l_R (as well as u_R if $u_R < \frac{1}{2}$) to $\frac{1}{2}$. $\square$

Theorem 8. *If the DM assigns weight $\alpha \leq \frac{3}{4}$ to $f \in \{L, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy COM, STR, and CON, and the DM isolates the three choice problems, then the relevant lower bound l_f is such that $l_f \leq \frac{1}{2}$.*

Proof. Without loss of generality assume $f = R$. Assume $l_R > \frac{1}{2}$ and we will reach a contradiction. The DM's selection of $\alpha R + (1-\alpha)L$ tells us $\alpha R + (1-\alpha)L \succeq R$ by COM, which means $\frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{1}{2} \succeq R$ by COM and CON, because letting $\beta = \frac{1}{4(1-\alpha)}$ it is evident $\beta(\alpha R + (1-\alpha)L) + (1-\beta)R = \frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{1}{2}$. So, $\frac{1}{2}R + \frac{1}{2}x \succ R$ for $x > \frac{1}{2}$ by STR, and the DM could strictly benefit from reducing l_R to $\frac{1}{2}$. $\square$

Theorem 9. *If the DM assigns weight $\alpha \geq \frac{3}{4}$ to $f \in \{L, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy COM, STR, CON, and WCI, and the DM isolates the three choice problems, then $u_f \geq \frac{1}{2}$.*

Proof. Without loss of generality assume $f = R$. COM, CON, and the DM's choice of how to randomize over L and R tells us $\frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{1}{2} \succeq \frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$. Then, STR and WCI tell us $\frac{1}{2}R + \frac{1}{2}z \succ z$ for $z \in [0, \frac{1}{2})$, and thus the agent can strictly benefit by increasing u_R to $\frac{1}{2}$ if $u_R < \frac{1}{2}$. $\square$

Theorem 10. *If the DM assigns weight $\alpha \leq \frac{1}{2}$ to $f \in \{L, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy COM, STR, CON, and WCI, and the DM isolates the three choice problems, then $u_f \leq \frac{1}{2}$.*

Proof. Without loss of generality assume $f = R$. COM, CON, and the DM's choice of how to randomize over L and R tells us $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2} \succeq \frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{1}{2}$, $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2} \succeq R$, and $\frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{1}{2} \succeq R$. Then, STR and WCI tell us $z \succ \frac{1}{2}R + \frac{1}{2}z$, $z \succ R$, and $\frac{1}{2}R + \frac{1}{2}z \succ R$, for $z \in (\frac{1}{2}, 1]$, and thus the agent can strictly benefit by decreasing u_R (and l_R if $l_R > \frac{1}{2}$) to $\frac{1}{2}$ if $u_R > \frac{1}{2}$. □

Theorem 11. *If the DM assigns weight α to $f \in \{L, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy COM, STR, and WCI, and the DM isolates the three choice problems, then if $\alpha = \frac{3}{4}$ it must be that $u_f \geq \frac{1}{2}$, while if $\alpha = \frac{1}{2}$ it must be that $u_f \leq \max(l_f, \frac{1}{2})$.*

Proof. Assume without loss that $f = R$. If $\alpha = \frac{3}{4}$ then COM tells us $\frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{1}{2} \succeq \frac{1}{2}R + \frac{1}{2}L = \frac{1}{2}$, so STR and WCI tells us for $x < \frac{1}{2}$ that $\frac{1}{2}R + \frac{1}{2}x \succ x$, so if $u_f < \frac{1}{2}$ the DM could do strictly better by increasing u_f to $\frac{1}{2}$. If $\alpha = \frac{1}{2}$ then COM tells us $\frac{1}{2}R + \frac{1}{2}L = \frac{1}{2} \succeq \frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{1}{2}$, so STR and WCI tells us for $x > \frac{1}{2}$ that $x \succ \frac{1}{2}R + \frac{1}{2}x$, so if $u_f > \max(l_f, \frac{1}{2})$ the DM could do strictly better by decreasing u_f to $\max(l_f, \frac{1}{2})$. □

Theorem 12. *If the preferences of the DM satisfy WP, COM, WSTR, CON, TR, and CI, and the DM isolates the three choice problems, then $l_L = u_L$ and $l_R = u_R$.*

Proof. Suppose the DM responds to the probability equivalent question about $f \in \{L, R\}$ with $u_f > l_f$. WP, COM, TR, CI, and the fact the the DM does not wish to decrease u_f tells us that for small $\epsilon > 0$ such that $u_f - \epsilon > l_f$ there is $z \in [u_f - \epsilon, u_f]$ such that $\frac{1}{2}f + \frac{1}{2}z \succ z = \frac{1}{2}z + \frac{1}{2}z \Rightarrow f \succ z$ because if not $u_f - \epsilon \succeq \frac{1}{2}f + \frac{1}{2}(u_f - \epsilon) \Rightarrow \frac{1}{2}(u_f - \epsilon) \succeq \frac{1}{2}f$ and thus for all $y \in (u_f - \epsilon, u_f]$ we have $\frac{1}{2}y \succ \frac{1}{2}f \Rightarrow y \succ \frac{1}{2}f + \frac{1}{2}y$ and the DM could do strictly better by lowering u_f. COM and the fact the the DM does not wish to increase l_f tells us that for all $\epsilon > 0$ there is $x \in [l_f, l_f + \epsilon]$ such that $\frac{1}{2}x + \frac{1}{2}f \succeq f$, so TR tells us $\frac{1}{2}x + \frac{1}{2}f \succ z$, and taking ϵ to zero COM and WSTR thus tell us $u_f < 1$. Thus, COM and the fact the the DM does not wish to increase u_f tells us that for all $\epsilon > 0$ there is $q \in [u_f, u_f + \epsilon]$ such that $q \succeq \frac{1}{2}f + \frac{1}{2}q$ and thus TR and WCI tell us $\frac{1}{2}q \succeq \frac{1}{2}f \Rightarrow \frac{1}{2}q + \frac{1}{2}x \succeq \frac{1}{2}f + \frac{1}{2}x \succeq f \succ z \Rightarrow \frac{1}{2}q + \frac{1}{2}x \succ z$, which contradicts WP as ϵ goes to zero. □

B Additional Experimental Details and Robustness Checks (for online publication)

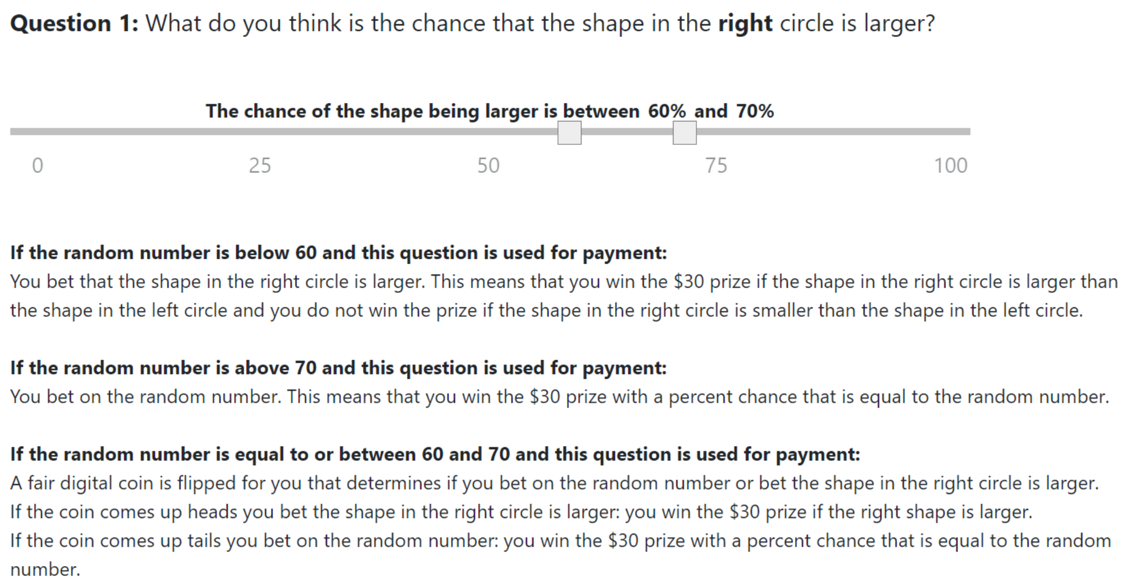
Subjects were invited to participate and register for the experiment using ORSEE (Greiner, 2015), and all sessions were done remotely between November 21st and December 2nd 2021. Subjects registered for a particular 10 hour time slot and had to start the experiment before the deadline. Before starting the experiment, subjects were required to read and sign a consent waiver.

For each type of round, Big-Shape and Lottery, subjects are first trained on the interface before completing a quiz with 5 questions: some are multiple choice questions and others involve subjects correctly using the interface sliders, in order to verify their understanding of the interface. If a subject finishes a quiz they receive \$2.50 (these are Canadian dollars, as with all other dollar amounts referred to in this paper) and, further, for each quiz question they answer correctly on the first attempt they are rewarded with an additional \$0.50. There was thus a maximum bonus payment of \$2.50+\$2.50 per quiz, or \$10 in total across quizzes. Subjects who do not answer a quiz question correctly are not able to advance to the next question without first providing a correct response. Further, in the multiple choice questions the answers are randomly permuted each time the question is answered incorrectly, so the subject cannot simply guess the possible answers in order. We observe how many attempts it takes each subject to answer each quiz question correctly. In general, subjects invested a great deal of time into the training with a median time spent on the two trainings and quizzes of 37.5 minutes. The training and quiz questions can be found in the supplementary materials.

In total, 250 subjects completed at least one quiz, and out of these subjects 218 ended up completing the entire experiment (both quizzes and all 24 rounds of decision problems). The 218 subjects that completed the entire experiment constitute the dataset we use for analysis.

The attrition rate does not seem concerning given that the experiment was conducted on-line during a busy part of the semester and subjects had the option to exit the experiment at the bottom of the screen in every round. During the training subjects are made aware of this feature of the experiment, and are informed that if they exit before the round with the question that was randomly selected to determine if they won the prize then they will not win the prize, but would still receive any

Figure 6: Graphical Interface for "Probability Equivalent" Problems



An example of a "probability equivalent" problem. Note that the text below the double-sliders updates automatically when the sliders are moved, and explains to the participant the payoff consequences of their choices.

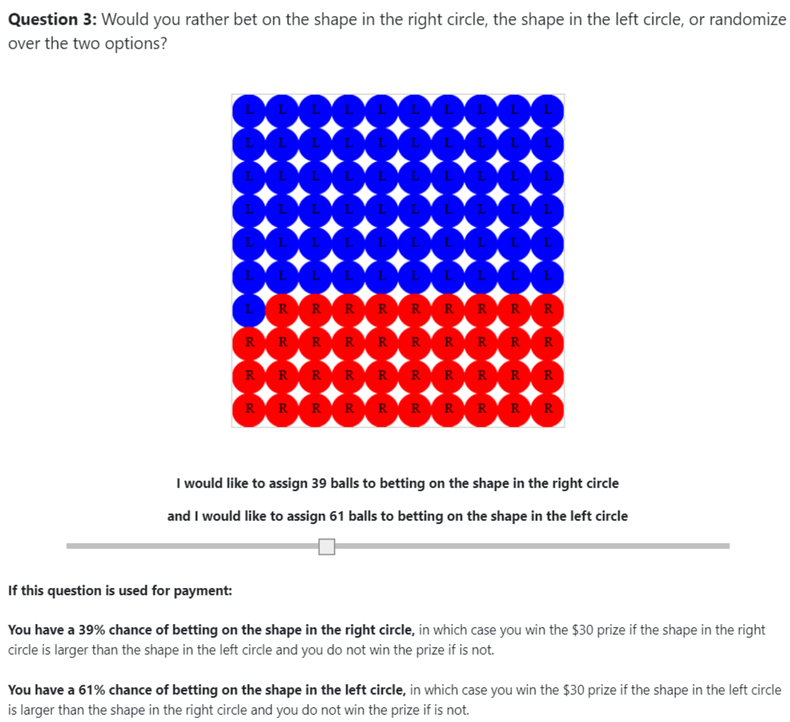
payments they earned from their performance on a quiz, and if they answered the question that was randomly selected to determine if they won the prize before they exited then choosing to exit would not impact whether or not they won the prize. We gave the subjects this forgiving exit option because we want their choices to be indicative of their preferences, not simply a bi-product of them trying to finish the experiment so that whatever benefit they have already earned would not be lost from exiting.

Subjects that did not finish the first quiz are also not very concerning. Subjects signed up for sessions several days in advance and knew that if they did not attempt the experiment then it could negatively impact their ability to participate in future experiments, so they may have started the experiment just to avoid being excluded from future experiments, or realized that the experiment was a larger time commitment than they had anticipated and stopped before the investment of time required to get through the initial training. Also, remember that subjects could not progress past a quiz question without first answering it correctly, which may have encouraged some of the subjects that did not understand the experimental design as well to quit.

Screenshots of the interface used in the Big-Shape rounds are provided in Figures

6 and 7. The initial position of the double-sliders for the probability equivalent questions are 0 and 100 for the lower and upper bound respectively. The initial position of the slider for the randomization question is uniformly distributed over $\{0, 1, \ldots, 100\}$ and is recorded in our data.

Figure 7: Graphical Interface for Randomization Problem



In order to visualize the randomization between betting on the left shape or the right shape, the participant chooses a composition of an urn that contains 100 balls. A randomly chosen ball determines the bet. Note that the participant can choose to bet on the right or the left shape (for sure) by choosing all red or all blue balls. The dynamically changing text below the urn explains the implications of choice to the participant.

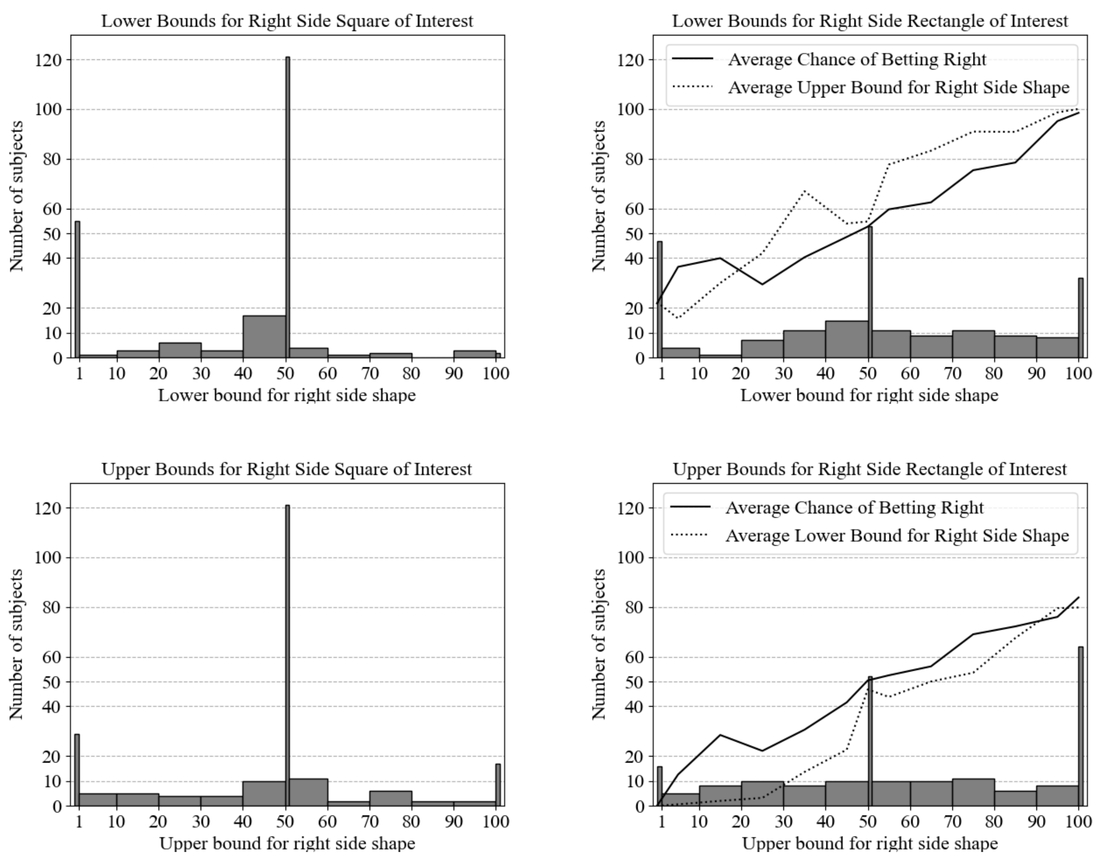
We collected complete process data for all 5 sliders in each round. We see when each change of each slider is made, and from what position to what position it is moved.

B.1 More on Squares vs Rectangles

As discussed in Section 5.5, there are two sets of choice problems that are particularly helpful for understanding when one should expect behavior that is inconsistent with the generalized VP and generalized SAAP models, and hence suggests incompleteness of preferences: the **rectangles of interest** and the **squares of interest**. As is argued in Section 5.5, identifying which shape is larger is equally challenging, but the rectangles of interest simulate a two-dimensional decision problem whereas the squares of interest represent a one-dimensional decision problem.

Subjects are much more likely to mix with exactly 50% chance over betting on the left and right-hand side shape when faced with the squares of interest compared to the rectangles of interest: 137 subjects do so when faced with the squares of interest and only 52 do so when faced with the rectangles of interest.

Figure 8



Subjects are much more likely to pick upper and lower bounds of 50 for the right-hand side shape when faced with the squares of interest compared to the rectangles of interest.

Figure 8 demonstrates subjects' choices of upper and lower bounds for the right-hand side shape when faced with the squares (the two plots on the left-hand side of Figure 8) and rectangles of interest (the two plots on the right-hand side of Figure

8). In Figure 8, the horizontal axes reflect the choices of bounds of subjects and has bins for subjects that chose exactly, 0, 50, or 100, and bins for subjects that chose in each group of 10 (i.e 10 to 19) barring choices of 0, 50, or 100. The vertical axes in Figure 8 indicate the number of subjects that chose bounds in each bin. Further, the dotted line in the plots on the right-hand side of Figure 8 indicate the average choices of the other bound for the interval in the round where subjects faced the rectangles of interest and demonstrate that the upper and lower bounds for the right-hand side rectangle of interest tend to move together, i.e. a low lower bound tends to correspond to a low upper bound as well. The solid line in the plots on the right-hand side of Figure 8 indicate the average choices of the chance of betting on the right-hand side shape in the round with the rectangles of interest and demonstrate that subjects' bets on the shapes in the randomization question tend to correspond to their "beliefs" about which shape is bigger, i.e. when the upper and lower bounds on the chance of the right-hand side shape being larger are high subjects tend to bet more on the right-hand side shape.[39]

B.2 Integration Data

In an attempt to get the subjects to respond to each of the three questions in each Big-Shape round (the question that asks how they would like to randomize over betting on the left and right-hand shape and the two probability equivalent questions that ask the chance of each shape being larger) as if it was the only question they were facing subjects are told that one question has been randomly selected before they start answering questions, and it is only the randomly selected question that determines their chance of winning the prize. Even though we do this, it is possible that subjects "integrate" across the questions, and for a particular random lottery r they behave as if they consider the different questions simultaneously and "hedge" their bets across the questions (Baillon et al., 2022b). In Section 3.2 we explore the theoretical implications for such subjects when the Variational Preferences (VP) model (Maccheroni et al., 2006) is imposed, and find necessary conditions that can be rejected by our observable behavior.

Can a combination of integration and the VP model explain our data? In short,

[39] The solid black and dotted black curves that are featured on the right-hand side of Figure 8 are withheld on the left-hand side of Figure 8 because the fewer observations at locations other than 50 cause the curves to look even noisier and, as such, are less interesting.

no, not well at least. If we look for subjects that violate Theorem 3 in at least one of the Big-Shape rounds we find that 65% (142/218) of subjects do so, if we look for subjects that violate Theorem 3 in at least three of the Big-Shape rounds we find that 46% (100/218) of subjects do so, and if we look for subjects that violate Theorem 3 in at least five of the Big-Shape rounds we find that 32% (69/218) of subjects do so.

If we worry some subjects are integrating and others are isolating, we can look for subjects that violate the generalized VP model under isolation (Theorem 2, Theorem 7, Theorem 8, Theorem 9, Theorem 10) in at least one round, and also violate Theorem 3 in at least one round, and we find 62% (136/218) of subjects do so. If we instead look for subjects that violate the generalized VP model under isolation in at least three rounds, and also violate Theorem 3 in at least three rounds, we find 43% (94/218) of subjects do so.

If we try to separate the double-slider behavior of the different subjects into "types" then the double-slider behavior of 18% of subjects (40/218) is consistent with our results on both isolation and integration (do not violate Theorem 2 or Theorem 3 in any round) and would include anyone that behaves in line with standard expected utility, the double slider behavior of 17% (36/218) of subjects is consistent with our results on integration but not isolation (violate Theorem 2 in a round, but do not violate Theorem 3 in any round), the double-slider behavior of 5% (10/218) of subjects is consistent with our results on isolation but not integration (do not violate Theorem 2 in any round, but do violate Theorem 3 in a round), the remaining 61% (132/218) of subjects violate our results on both integration and isolation with their double-sliders (violate Theorem 2 in a round and Theorem 3 in a round).

Further evidence that seems to support that "integrating" is relatively rare can be found in Section 5.4.

B.3 Reduction of Compound Lottery Data

The definition we use of a mixture of two acts $\alpha f+(1-\alpha)g$ in the third paragraph of Section 2 is of statewise-mixture. Given the experimental evidence that suggests an empirical association between ambiguity sensitivity and failure to reduce compound lotteries, it is important to pin down how sensitive our results are to the latter. One of the advantages of the axiomatic approach employed in the proofs contained in the current study is that it is evident where reduction of compound lotteries (ROCL) may play a role in them, and the potentially most problematic way it may affect our identification results. In particular, we assume that equal weights assigned to betting on the left and right shape are equivalent to these weights being assigned to a 50% chance of winning. Thus, randomizing equally over betting on the left and right shape is equivalent to a 50% chance of winning, and a three quarter chance of betting on the right shape and a one quarter chance of betting on left shape is equivalent to a one half chance of betting on the right shape and a one half chance of winning with a 50% chance (because half the time the DM is betting on the right shape and the other half of the time they are randomizing equally over the left and right shape).

This is a very specific form of ROCL that is relatively simple, and though ROCL might be a problematic assumption in some instances, our data indicates that this specific form is not the main source of the apparent incompleteness that we observe. First, several of the quiz questions feature compound lotteries and thus it would seem that subjects that perform better on the quiz should be more likely to be reducing compound lotteries, but as is evident from Table 8, it is not subjects with low quiz scores that are driving apparent incompleteness. Second, one of our Big-Shape rounds does not have an image in it and this creates a compound lottery that is particularly relevant to the compound lottery where the agent randomizes with equal chances over betting on the left and right shape. This follows since when one randomizes over betting on the left and right shape with equal chances you have a 50% chance of winning in every state of the world, and in the round with no image there is a 50% chance of winning no matter the shape you bet on. Thus, if a subject reduces compound lotteries a standard model would predict that in the round with no image they should choose a degenerate interval of 50 in both PE questions. 70% of our subjects do exactly this, which indicates that most subjects do seem to be reducing simple compound lotteries. Further, if we restrict the analysis to those subjects that choose a degenerate interval in both PE questions in the Big-Shape round with no image – then 80% of such subjects have at least one round in which they violate the generalized VP model, 59% of such subjects have at least three rounds in which they violate the generalized VP model, and 45% of such subjects have at least five rounds in which they violate the generalized VP model. Thus, we still observe substantial behavior that rejects quite flexible models of complete preferences even when we restrict analysis to subjects that appear to reduce the compound lottery in the round with no image.

B.4 Fatigue and Mistakes Data

It is possible that as subjects fatigue they begin to make mistakes, and these mistakes are causing the complete models of preferences we consider to be rejected, but our data does not support this hypothesis. It seems that subjects are really careful when using the sliders. It is clear from the left-hand side of Figure 8 that subjects are making very few mistakes when trying to put their slider on 50. Further, some subjects were randomly assigned to face the Big-Shape rounds first and then face the 12 Lottery rounds after (see Section 4), and the rest of subjects face the 12 Lottery rounds first and then face the 12 Big-Shape rounds after (denote these subjects by BSL). If fatigue is the cause of complete models of preference being rejected then we should expect that the subjects that face the 12 Big-Shape rounds at the end of their session should be more likely to reject the complete models of preferences we study, and yet as Table 6, the order in which they face the different rounds seems irrelevant to our main conclusions. In Table 6 the last column features the subjects that had the Big-Shape rounds at the end of their session (BSL), and the percent of subjects that cause rejections do not seem to be systematically impacted.

Table 6: Number of Subjects that Violate Models Conditional on Order

	All	BSL
Number of subjects (NOS)	218	96
NOS that violate generalized VP model	183 (84%)	79 (82%)
NOS that violate generalized VP model 3 or more times	144 (66%)	61 (64%)
NOS that violate generalized VP model 5 or more times	119 (55%)	49 (51%)
NOS that violate generalized VP without CON	99 (45%)	38 (40%)
NOS that violate generalized SAAP	159 (73%)	71 (74%)
NOS that violate generalized SAAP 3 or more times	92 (42%)	38 (40%)
NOS that violate generalized UAP	81 (37%)	37 (39%)

The last column contains analysis on the subset of subjects that faced the Big-Shape rounds after the Lottery rounds.

We can also raise the threshold for rejecting behavior and still get substantial rejections. For instance, if we look for upper bounds that sum to a threshold of 105 or more (instead of the threshold for rejection of 102 in rounds with non-degenerate intervals and 103 in rounds with only degenerate intervals imposed by Theorem 2, see the discrete version of the result in the supplementary materials) we still get that 77% of subjects have at least one of such rounds, 57% of subjects have at least 3 of such rounds, and 46% of subjects have at least 5 of such rounds. If we look for upper bounds that sum to a threshold of 110 or more we still get that 74% of subjects have at least one of such rounds, 51% of subjects have at least 3 of such rounds, and 37% of subjects have at least 5 of such rounds.

B.5 Subjects That Passed All Three Sanity Checks

We can also restrict the analysis to subjects that performed "perfectly" on all three sanity checks. A subject is said to **perform perfectly on all three sanity checks** if when they face the Big-Shape round with no image they submit $l_L = u_L = l_R = u_R = 50$, when they face the Big-Shape round with Image 5 (in Figure 9 in the supplementary materials) they submit $l_L = u_L = 100$, $l_R = u_R = 0$, and bet on L with a 100% chance, and when they face the Big-Shape round with Image 11 (also in Figure 9 in the supplementary materials) they submit $l_L = u_L = 0$, $l_R = u_R = 100$, and bet on R with a 100% chance. The analogue of Table 3 when analysis is instead restricted to such subjects can be found in the last column of Table 7.

Table 7: Subjects that Violate Models Conditional on Passing Sanity Checks

	All	Passed three sanity checks
Number of subjects (NOS)	218	191
NOS that violate generalized VP model	183 (84%)	87 (76%)
NOS that violate generalized VP model 3 or more times	144 (66%)	62 (54%)
NOS that violate generalized VP model 5 or more times	119 (55%)	45 (39%)
NOS that violate generalized VP without CON	99 (45%)	41 (36%)
NOS that violate generalized SAAP	159 (73%)	71 (62%)
NOS that violate generalized SAAP 3 or more times	92 (42%)	34 (30%)
NOS that violate generalized UAP	81 (37%)	22 (19%)

The last column contains analysis on the subset of subjects that passed all three sanity checks.

Overall, 52% (114/218) of our subjects performed perfectly on all three sanity checks and, surprisingly, this subset of subjects still produce pervasive violations of the generalized VP model Maccheroni et al. (2006). If we look for the proportion of subjects that passed all three sanity checks but also violate the generalized VP model under isolation (Theorem 2, Theorem 7, Theorem 8, Theorem 9, or Theorem 10), in the Big-Shape rounds we find that 76% (87/114) of them do so in at least one round, 54% (62/114) of them do so in at least three rounds, and 39% (45/114) of them do so in at least five rounds.

Subjects that passed the three sanity checks are a bit less likely to report an interval in a round, reporting a non-degenerate interval in 29% of their rounds, and report slightly smaller but still large intervals, with an average length of 19 when an interval is reported. These subjects randomize over L and R in 58% of their rounds and randomize over L and R 94% of the time if their response in the PE questions reject Theorem 2 in a round.

B.6 Controlling For Quiz Performance

As is mentioned in the body of the paper, it is not lack of understanding on the part of subjects that is driving our estimates of incomplete preferences. In Table 8 the second column has all subjects, the third column has all subjects that got 4 or more on the quiz (the passing grade for undergraduate students at the University of Surrey), and the fourth column has subjects that got 7 or more on the quiz (the passing grade for graduate students at the University of Toronto). Table 8 indicates that the proportion of subjects that contradict our different necessary conditions is essentially unchanged when we remove subjects that did not perform as well on the quiz.

Table 8: Number of Subjects That Violate Models Conditional on Quiz Score

	All	Quiz ≥ 4	Quiz ≥ 7
Number of subjects (NOS)	218	191	99
NOS that violate generalized VP model	183 (84%)	159 (83%)	81 (82%)
NOS that violate generalized VP model 3 or more times	144 (66%)	125 (65%)	63 (64%)
NOS that violate generalized VP model 5 or more times	119 (55%)	102 (53%)	52 (53%)
NOS that violate generalized VP without CON	99 (45%)	84 (44%)	38 (38%)
NOS that violate generalized SAAP	159 (73%)	138 (72%)	66 (67%)
NOS that violate generalized SAAP 3 or more times	92 (42%)	76 (40%)	35 (35%)
NOS that violate generalized UAP	81 (37%)	65 (34%)	34 (34%)

The last and second last column contain analysis on the subset of subjects that got a combined score of 7 or more out of 10 on the two quizzes and 4 or more out of 10 on the two quizzes, respectively.

C Supplementary Materials (not for publication)

C.1 Discrete Interval and Isolation Results

This subsection and the next present amended results that accommodate the discrete nature of the double-sliders in the experiment discussed by Halevy, Walker-Jones, and Zrill (2023). If the continuous version of a result is called Theorem Y in the work of Halevy et al. (2023), then the discrete version found here is called Theorem Y.1. Similarly, the discrete version of Corollary 1 in the work of Halevy et al. (2023) is presented here as Corollary 1.1.

Remember: Halevy et al. (2023) study a model with a set of binary **states** S, a convex set X of **consequences**, and **(simple) acts** $\mathcal{F}$, which are measurable functions from S to X. For every $x \in X$, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. Acts always result in the DM receiving one of two payments, either $m > 0$, or nothing. To match the set of possible outcomes for the sliders in the experiment, we thus, as a minor abuse of notation, change the set of consequences X (from the one considered by Halevy et al. (2023)) to be $X = [0, 100]$, but use the same incentivization of questions as is described in Section 2.1 from the work of Halevy et al. (2023), with $x \in X$ now representing the constant act that produces a x% chance of winning the monetary prize m in each of the two states.

To ease the connection to the experimental implementation, we consider two special acts $L, R \in \mathcal{F}$ (short for "bet on left" and "bet on right"). The acts L and R are **bets on the two possible states**: a correct bet wins the DM a payment of $m > 0$, and nothing otherwise.

Correspondingly, let $l_L, u_L \in \{0, 1, \ldots, 100\}$ with $l_L \leq u_L$ denote the lower and upper bound reported by the DM for the left shape, and let $l_R, u_R \in \{0, 1, \ldots, 100\}$ with $l_R \leq u_R$ denote the lower and upper bound reported by the DM for the right shape. Further, the random lottery r is now distributed over $[0, 100]$, and we relax the assumption that the DM assigns a positive probability to each open interval of potential r, and instead only assume that the DM assigns a positive probability to each interval of potential r that contains an integer. This means that our results are robust to subjects in the experiment having strange beliefs about how the random lottery is drawn.

Our results in this section only produce a rejection of the axioms if the DM can change their response to one of the questions and do weakly better for all r and strictly better for at least one integer $r \in \{0, 1, \ldots, 100\}$. To define this notion formally, for each $r \in [0, 100]$ let $Q(r) \in \mathcal{F}$ denote the act assigned to the DM when the question they are answering is used for payment and the random lottery is r. We then impose, given some sub-set of axioms on their preferences, that the DM's answer is not such that there is an alternative way of answering the question that would result in alternate acts $\tilde{Q}(r) \in \mathcal{F}$ for each $r \in [0, 100]$ if the question is used for payment such that $\exists\, r \in \{0, 1, \ldots 100\}$ with: $\tilde{Q}(r) \succ Q(r)$, and $\forall\, r \in [0, 100]$: $\tilde{Q}(r) \succeq Q(r)$. The results below use the axioms from Halevy et al. (2023), which are provided here for the reader's convenience.

Axiom 0 (Winning is Preferred (WP)).

For all $x, y \in X$, if $y > x$ then $y \succ x$.

Axiom 1 (Completeness (COM)).

For all $f, g \in \mathcal{F}$, either $f \succeq g$ or $g \succeq f$.

Axiom 2 (Weak Statewise Transitivity (WSTR)).

For all $f, g, h \in \mathcal{F}$, if h statewise dominates f and $f \succeq g$ then $h \succeq g$, and if h statewise dominates f and $g \succeq h$ then $g \succeq f$.

Axiom 2′ (Statewise Transitivity (STR)).

For all $f, g, h \in \mathcal{F}$, if h statewise dominates f and $f \succeq g$ then $h \succ g$, and if h statewise dominates f and $g \succeq h$ then $g \succ f$.

Axiom 3 (Convexity (CON)).

For all $f, g, h \in \mathcal{F}$, if $f, h \succeq g$, then for all $\alpha \in (0, 1)$: $\alpha f + (1 - \alpha)h \succeq g$.

Axiom 4 (Transitivity (TR)).

For all $f, g, h \in \mathcal{F}$, if $f \succeq g$ and $g \succeq h$, then $f \succeq h$.

Axiom 5 (Weak Certainty Independence (WCI) (Maccheroni et al., 2006)).

For all $f, g, \in \mathcal{F}$, if $x, y \in X$, then for all $\alpha \in (0, 1)$:

$$\alpha f + (1 - \alpha)x \succeq \alpha g + (1 - \alpha)x \implies \alpha f + (1 - \alpha)y \succeq \alpha g + (1 - \alpha)y.$$

Axiom 5′ (Certainty Independence (CI) (Gilboa & Schmeidler, 1989)).

For all $f, g, \in \mathcal{F}$, $x \in X$, and $\alpha \in (0, 1)$:

$$f \succ g \iff \alpha f + (1 - \alpha)x \succ \alpha g + (1 - \alpha)x.$$

Theorem 1.1. If the preferences of the DM satisfy WP, COM, WSTR, and CON, and the DM isolates the three choice problems, then their upper bounds u_L and u_R are such that:

$$\frac{u_L + u_R - 1}{4} + 25 \geq \min\{u_L,\, u_R\} - 1.$$

Proof. First of all, if the inequality does not hold then $u_L,\, u_R > 0$, and for $f \in \{L,\, R\}$ there is $z \in [u_f - 1, u_f]$ such that $\frac{1}{2}f + \frac{1}{2}z \succeq z$. To see why, we will break things down into cases. If the associated lower bound l_f is such that $l_f < u_f$, then, given COM, the DM could do strictly better by reducing their upper bound by one unless there is $z \in (u_f - 1, u_f]$ such that $\frac{1}{2}f + \frac{1}{2}z \succeq z$. If $l_f = u_f$, then, given COM, there is $z \in [u_f - 1, u_f]$ such that $f \succeq \frac{1}{2}f + \frac{1}{2}z$, otherwise the DM could do strictly better by reducing their lower bound, and by WSTR we can then infer $f \succeq \frac{1}{2}f + \frac{1}{2}(u_f - 1)$. If $f \succ \frac{1}{2}f + \frac{1}{2}(u_f - 1)$, then COM and CON tell us $\frac{1}{2}f + \frac{1}{2}(u_f - 1) \succeq u_f - 1$. If instead $f \sim \frac{1}{2}f + \frac{1}{2}(u_f - 1)$ and further $u_f - 1 \succ f$, then it must be $\frac{1}{2}f + \frac{1}{2}u_f \succeq u_f$ otherwise the DM could do strictly better by lowering both bounds by one by COM and WSTR. So, if $f \sim \frac{1}{2}f + \frac{1}{2}(u_f - 1)$ then either $\frac{1}{2}f + \frac{1}{2}u_f \succeq u_f$ or $f \succeq u_f - 1$, and in the latter case COM and CON then tell us $\frac{1}{2}f + \frac{1}{2}(u_f - 1) \succeq u_f - 1$. Thus, there is $z \in [u_f - 1, u_f]$ such that $\frac{1}{2}f + \frac{1}{2}z \succeq z$.

Now, for $f,\, g \in \{L,\, R\}$ with $f \neq g$, let z_f denote a constant act in $[u_f - 1, u_f]$ such that $\frac{1}{2}f + \frac{1}{2}z_f \succeq z_f$ and let z_g denote a constant act in $[u_g - 1, u_g]$ such that $\frac{1}{2}g + \frac{1}{2}z_g \succeq z_g$. It is without loss to assume $z_f \geq z_g$ and $u_f \geq u_g$. It is then the case that $\frac{1}{2}f + \frac{1}{2}z_f \succeq z_g$ by WSTR. CON then tells us $\frac{1}{4}L + \frac{1}{4}R + \frac{1}{4}z_f + \frac{1}{4}z_g \succeq z_g$, but $\frac{1}{4}L + \frac{1}{4}R + \frac{1}{4}z_f + \frac{1}{4}z_g = \frac{1}{2}50 + \frac{z_f + z_g}{4} = 25 + \frac{z_f + z_g}{4}$, so WP tells us $25 + \frac{z_f + z_g}{4} \geq z_g \Rightarrow \frac{u_f + z_g}{4} + 25 \geq z_g \Rightarrow \frac{u_f + u_g - 1}{4} + 25 \geq u_g - 1.$ ■

Theorem 2.1. If the DM's preferences satisfy WP, COM, CON, and WCI, and the DM isolates the three choice problems, then $u_L + u_R \leq 101$ unless: $u_L + u_R = 102$, $u_L = l_L$, and $u_R = l_R$.

Proof. Assume $u_L + u_R \geq 102$. The upper bounds, COM, and CON, imply (since the DM would not strictly benefit from lowering either upper bound and $u_L + u_R \geq 102 \Rightarrow \min(u_L,\, u_R) \geq 2$) that there is $z_R \in [u_R - 1,\, u_R]$ and $z_L \in [u_L - 1,\, u_L]$ such that:

$$\frac{1}{2}z_R + \frac{1}{2}R \succeq z_R \text{ and } \frac{1}{2}z_L + \frac{1}{2}L \succeq z_L.$$

This is evident if neither interval is degenerate, but it is also true if there is a degenerate interval since COM and CON imply:

$$z_R \succ \frac{1}{2}z_R + \frac{1}{2}R \Rightarrow z_R \succ R,\ \frac{1}{2}z_R + \frac{1}{2}R \succeq R,$$

$$\text{and } z_L \succ \frac{1}{2}z_L + \frac{1}{2}L \Rightarrow z_L \succ L,\ \frac{1}{2}z_L + \frac{1}{2}L \succeq L,$$

and otherwise the DM could do strictly better by lowering both bounds of the degenerate interval by one. Thus, WCI tells us for all such z_R and z_L:

$$\frac{1}{2}z_L + \frac{1}{2}R \succeq \frac{1}{2}z_R + \frac{1}{2}z_L \text{ and } \frac{1}{2}z_R + \frac{1}{2}L \succeq \frac{1}{2}z_L + \frac{1}{2}z_R.$$

But, then CON tells us for all such z_R and z_L:

$$\frac{1}{4}L + \frac{1}{4}R + \frac{1}{4}z_R + \frac{1}{4}z_L \succeq \frac{1}{2}z_R + \frac{1}{2}z_L.$$

Notice that $\frac{1}{4}L(s) + \frac{1}{4}R(s) = 25$ for all $s \in S$, so $\frac{1}{4}L + \frac{1}{4}R = 25 \in X$. Thus, WP tells us:

$$\frac{1}{4}L + \frac{1}{4}R + \frac{1}{4}z_R + \frac{1}{4}z_L = \frac{1}{2}\left(\frac{100}{2}\right) + \frac{1}{2}\left(\frac{1}{2}z_R + \frac{1}{2}z_L\right) \geq \frac{1}{2}z_R + \frac{1}{2}z_L$$

$$\Rightarrow z_R + z_L \leq 100 \Rightarrow u_R + u_L = 102 \text{ (since } u_R + u_L \geq 102) \Rightarrow z_R + z_L = 100.$$

Thus, since for all z_R and z_L such that:

$$\frac{1}{2}z_R + \frac{1}{2}R \succeq z_R \text{ and } \frac{1}{2}z_L + \frac{1}{2}L \succeq z_L,$$

we know $z_R + z_L = 100$, COM tells us it must be that for all $z_R \in (u_R - 1,\, u_R]$ and $z_L \in (u_L - 1,\, u_L]$ that:

$$z_R \succ \frac{1}{2}z_R + \frac{1}{2}R \text{ and } z_L \succ \frac{1}{2}z_L + \frac{1}{2}L,$$

and therefore $l_L = u_L$ and $l_R = u_R$, because otherwise the DM could do strictly better by lowering an upper bound. ■

Corollary 1.1. If the DM's preferences satisfy WP, COM, CON, and WCI, $l_L < u_L$ or $l_R < u_R$, and the DM isolates the three choice problems, then:

$$b_{av} \leq 50.5 - \frac{1}{2}s_{av}.$$

Proof. If the DM's preferences satisfy WP, COM, CON, and WCI, and $l_L < u_L$ or $l_R < u_R$, then Theorem 2.1 tells us:

$$b_{av} = \frac{1}{2}(u_L + u_R) - \frac{1}{2}s_{av} \leq 50.5 - \frac{1}{2}s_{av}. \blacksquare$$

Theorem 5.1. If the DM assigns weight α to $f \in \{L,\, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy COM and STR, and the DM isolates the three choice problems, then if $\alpha = 1$ it must be that $l_f \geq \min(u_f,\, 50)$, while if $\alpha = \frac{3}{4}$ it must be that $l_f \leq 51$.

Proof. Assume without loss that $f = R$. If $\alpha = 1$ then COM tells us $R \succeq \frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{100}{2}$, so STR tells us for $x < 50$ that $R \succ \frac{1}{2}R + \frac{1}{2}x$, so if $l_f < \min(u_f,\, 50)$ the DM could do strictly better by increasing l_f to $\min(u_f,\, 50)$. If $\alpha = \frac{3}{4}$ then COM tells us $\frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{100}{2} \succeq R$, so STR tells us for $x > 50$ that $\frac{1}{2}R + \frac{1}{2}x \succ R$, so if $l_f > 51$ the DM could do strictly better by decreasing l_f to 51. $\blacksquare$

Theorem 6.1. If the DM selects $f \in \{L,\, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy WP, COM, WSTR, CON, and TR, and the DM isolates the three choice problems, then $u_f \geq 49$.

Proof. Assume not, and $u_f < 49$. Without loss of generality assume $f = R$. The DM's selection of R tells us $R \succeq \frac{1}{4}L + \frac{3}{4}R = \frac{1}{2}\frac{100}{2} + \frac{1}{2}R$ by COM. So, $R \succeq \frac{1}{2}x + \frac{1}{2}R$, for $x \leq 50$ by WSTR. The DM's selection of R also tells us $R \succeq \frac{1}{2}L + \frac{1}{2}R = \frac{100}{2}$ by COM, thus $\frac{1}{2}\frac{100}{2} + \frac{1}{2}R \succeq \frac{100}{2}$ by COM and CON. Further, $R \succ x$, for $x < 50$ by WP and TR. But, then we have the desired contradiction since the DM can do strictly better by increasing both u_R and l_R to 50. $\blacksquare$

Theorem 7.1. If the DM selects $f \in \{L,\, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy COM, STR, and CON, and the DM isolates the three choice problems, then the relevant lower bound l_f is such that $l_f \geq 50$.

Proof. Assume not, and $l_f < 50$. Without loss of generality assume $f = R$. The DM's selection of R tells us $R \succeq \frac{1}{2}L + \frac{1}{2}R = \frac{100}{2}$ by COM, thus $\frac{1}{2}R + \frac{1}{2}\frac{100}{2} \succeq \frac{100}{2}$ by COM and CON, $R \succ x$ for $x < 50$ by STR, and the DM's selection of R also tells us $R \succeq \frac{1}{4}L + \frac{3}{4}R = \frac{1}{2}\frac{100}{2} + \frac{1}{2}R$ by COM, so, $R \succ \frac{1}{2}x + \frac{1}{2}R$, for $x < 50$ by STR. But, then the DM can do strictly better by increasing l_R (and perhaps u_R) to 50. $\blacksquare$

Theorem 8.1. If the DM assigns weight $\alpha \leq \frac{3}{4}$ to $f \in \{L, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy COM, STR, and CON, and the DM isolates the three choice problems, then the relevant lower bounds l_f is such that $l_f \leq 51$.

Proof. Assume not, and $l_f > 51$ for $f \in \{L, R\}$. Without loss of generality assume $f = R$. The DM's selection of $\alpha R + (1-\alpha)L$ tells us $\alpha R + (1-\alpha)L \succeq R$ by COM, which means $\frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{100}{2} \succeq R$ by COM and CON. So, $\frac{1}{2}R + \frac{1}{2}x \succ R$ for $x > 50$ by STR, and the DM could strictly benefit from reducing l_R to 51. ■

Theorem 9.1. If the DM assigns weight $\alpha \geq \frac{3}{4}$ to $f \in \{L, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy COM, STR, CON, and WCI, and the DM isolates the three choice problems, then $u_f \geq 49$.

Proof. Without loss of generality assume $f = R$. COM, CON, and the DM's choice of how to randomize over L and R tells us $\frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{100}{2} \succeq \frac{1}{2}R + \frac{1}{2}L = \frac{100}{2}$. Then, STR and WCI tell us $\frac{1}{2}R + \frac{1}{2}z \succ z$ for $z \in [0, 50)$, and thus the agent can strictly benefit by increasing u_R to 49 if $u_R < 49$. ■

Theorem 10.1. If the DM assigns weight $\alpha \leq \frac{1}{2}$ to $f \in \{L, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy COM, STR, CON, and WCI, and the DM isolates the three choice problems, then $u_f \leq 50$.

Proof. Without loss of generality assume $f = R$. COM, CON, and the DM's choice of how to randomize over L and R tells us $\frac{1}{2}R + \frac{1}{2}L = \frac{100}{2} \succeq \frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{100}{2}$, $\frac{1}{2}R + \frac{1}{2}L = \frac{100}{2} \succeq R$, and $\frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{100}{2} \succeq R$. Then, STR and WCI tell us $z \succ \frac{1}{2}R + \frac{1}{2}z$, $z \succ R$, and $\frac{1}{2}R + \frac{1}{2}z \succ R$ for $z \in (50, 100]$, and thus the agent can strictly benefit by decreasing u_R to 50 (and l_R if $l_R > 50$) if $u_R > 50$. ■

Theorem 11.1. If the DM assigns weight α to $f \in \{L, R\}$ when asked how they would like to randomize over L and R, their preferences satisfy COM, STR, and WCI, and the DM isolates the three choice problems, if $\alpha = \frac{3}{4}$ it must be that $u_f \geq 49$, while if $\alpha = \frac{1}{2}$ it must be that $u_f \leq \max(l_f, 50)$.

Proof. Assume without loss that $f = R$. If $\alpha = \frac{3}{4}$ then COM tells us $\frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{1}{2}\frac{100}{2} \succeq \frac{1}{2}R + \frac{1}{2}L = \frac{100}{2}$, so STR and WCI tells us for $x < 50$ that $\frac{1}{2}R + \frac{1}{2}x \succ x$, so if $u_f < 49$ the DM could do strictly better by increasing u_f to 49. If $\alpha = \frac{1}{2}$ then COM tells us $\frac{1}{2}R + \frac{1}{2}L = \frac{100}{2} \succeq \frac{3}{4}R + \frac{1}{4}L = \frac{1}{2}R + \frac{100}{2}$, so STR and WCI tells us for

$x > 50$ that $x \succ \frac{1}{2}R + \frac{1}{2}x$, so if $u_f > \max(l_f,\, 50)$ the DM could do strictly better by decreasing u_f to $\max(l_f,\, 50)$. ■

Theorem 12.1. If the preferences of the DM satisfy COM, STR, CON, TR, and CI, and the DM isolates the three choice problems, then $l_L \geq u_L - 1$ and $l_R \geq u_R - 1$.

Proof. Suppose the DM responds to the probability equivalent question about $f \in \{L,\, R\}$ with $u_f > l_f + 1$ (notice that this implies $u_f \geq 2$ and $l_f \leq 98$). COM, STR, CI, and the fact the the DM does not wish to decrease u_f tell us that $\frac{1}{2}f + \frac{1}{2}(u_f - 1) \succ u_f - 1 = \frac{1}{2}(u_f - 1) + \frac{1}{2}(u_f - 1) \Rightarrow f \succ u_f - 1$ because if not $u_f - 1 \succeq \frac{1}{2}f + \frac{1}{2}(u_f - 1) \Rightarrow \frac{1}{2}(u_f - 1) \succeq \frac{1}{2}f$ and then for all $y \in (u_f - 1,\, u_f]$ we have $\frac{1}{2}y \succ \frac{1}{2}f \Rightarrow y \succ \frac{1}{2}f + \frac{1}{2}y$ and the DM could do strictly better by lowering u_f. Similarly, COM, STR, and the fact the the DM does not wish to increase l_f tell us that $\frac{1}{2}(l_f + 1) + \frac{1}{2}f \succ f$. It cannot be that $f \succ 99.9$ because then we have a contradiction with COM, STR, TR, and $\frac{1}{2}(l_f + 1) + \frac{1}{2}f \succ f \succ 99.9$ because 99.9 statewise dominates $\frac{1}{2}(l_f + 1) + \frac{1}{2}f$ since $l_f + 1 \leq 99$. It cannot be that $0.1 \succeq f$ because then we have a contradiction with COM, STR, TR, and $0.1 \succeq f \succ u_f - 1$ because $u_f - 1$ statewise dominates 0.1 since $u_f - 1 \geq 1$. Thus, WP, COM, and TR, tell us there is $x \in (0,\, 100)$ such that if $y > x$ then $y \succ f$ and if $y < x$ then $f \succ y$, and thus using CI if $y > x$ then $\frac{1}{2}y \succ \frac{1}{2}f$ and $y \succ \frac{1}{2}f + \frac{1}{2}y$, and if $y < x$ then $\frac{1}{2}f \succ \frac{1}{2}y$ and $\frac{1}{2}f + \frac{1}{2}y \succ y$. Picking arbitrarily small $\epsilon > 0$, and using WP, TR, and CI, we thus have $\frac{1}{2}(x+\epsilon) \succ \frac{1}{2}(u_f - 1) \Rightarrow x \geq u_f - 1$ and $\frac{1}{2}(l_f + 1) + \frac{1}{2}(x+\epsilon) \succ x - \epsilon = \frac{1}{2}(x - 3\epsilon) + \frac{1}{2}(x+\epsilon) \Rightarrow \frac{1}{2}(l_f + 1) \succ \frac{1}{2}(x - 3\epsilon) \Rightarrow l_f + 1 \geq x$, so it must be that $x = u_f - 1 = l_f + 1$, but then the DM could do strictly better by lowering u_f to $u_f - 1$. ■

C.2 Discrete Interval and Integration Results

We now turn to our results that address the potential for the DM to be "integrating" (Baillon et al., 2022b) and answering multiple questions in conjunction instead of "isolating" and answering each question as if it is the only question they face. When we say the DM is answering a set of questions in conjunction we mean that for each random lottery they are trying to answer each of said questions in a way that maximizes their expected payoff given the weights that they think are the probabilities of each of said question being used for payment if one of the questions from the set is being used for payment. To define integration formally we need to introduce some new notation. For each random lottery $r \in [0,\, 100]$, let $Q_i(r) \in \mathcal{F}$

for $i \in \{1, 2, 3\}$ denote the act assigned to the DM when question i is used for payment and the random lottery is r. We then say the DM answers a subset of questions $\mathcal{Q} \subseteq \{1, 2, 3\}$ in **conjunction** with **weights** $\beta_j \geq 0$ for each $j \in \mathcal{Q}$ such that $\sum_{j\in\mathcal{Q}} \beta_j = 1$ if there is not an alternative way of answering the questions that would result in alternate acts $\tilde{Q}_j(r) \in \mathcal{F}$ for each $r \in [0, 100]$ and $j \in \mathcal{Q}$ such that $\exists\, r \in \{0, 1, \ldots, 100\}$ such that:

$$\sum_{j\in\mathcal{Q}} \beta_j \tilde{Q}_j(r) \succ \sum_{j\in\mathcal{Q}} \beta_j Q_j(r),$$

and $\forall\, r \in [0, 100]$:

$$\sum_{j\in\mathcal{Q}} \beta_j \tilde{Q}_j(r) \succeq \sum_{j\in\mathcal{Q}} \beta_j Q_j(r).$$

If we were to instead assume a particular distribution of r, for instance the uniform distribution over $\{0, 1, \ldots, 100\}$ that is actually used in the experiment, then rejection would be easier as we could reject the behavior of the DM if the they did not maximize their preference relation across the potential realizations of the random lottery and the question that is used for payment, and we are thus setting a relatively high bar for rejection of DM behavior. We also do not assume that $\beta_i = \frac{1}{3}\, \forall i \in \{1, 2, 3\}$ as is the reality in the experiment, but because of their symmetry we do assume in our results that the weights on the two probability equivalent (double-slider) questions are the same and strictly positive when the DM is answering them.

To ease exposition, we introduce Axiom 6 below, which is a feature of the models of Variational Preferences (Maccheroni et al., 2006) (axiom A.3 in their paper), Uncertainty Averse Preferences (Cerreia-Vioglio et al., 2011) (axiom A.5 in their paper), and Smooth Ambiguity Preferences (Klibanoff et al., 2005; Denti & Pomatto, 2022) (by Lemma 6 in the work of Denti and Pomatto (2022)). We then introduce three more simple lemmas, Lemma 4, Lemma 5, and Lemma 6, that build upon Axiom 6.

Axiom 6 (Continuity). If f, g, $h \in \mathcal{F}$, the sets $\{\alpha \in [0, 1] : \alpha f + (1 - \alpha)g \succeq h\}$ and $\{\alpha \in [0, 1] : h \succeq \alpha f + (1 - \alpha)g\}$ are closed.

Lemma 4. If the preferences of the DM satisfy WP, COM, WSTR, CON, TR, WCI, and Axiom 6, then they satisfy STR, and for all $\beta \in [0, 1]$, and e, $j \in \mathcal{F}$ there is a $w \in [0, 100]$ such that $\beta e + (1 - \beta)j \sim \beta w + (1 - \beta)j$, and if $\beta > 0$ then such w is unique.

Proof. First, notice that Proposition 1 from the work of Halevy et al. (2023) tells us STR holds since, given WP, it is trivial to show that Axiom 1 (COM) through Axiom 6 imply axioms A.1 through A.6 from the model of Maccheroni et al. (2006), the only one that is perhaps not trivial is axiom A.4, but it is not hard to show it is implied by WSTR and Axiom 6.

Let $x = 100 \in X$ and $y = 0 \in X$. If $\beta = 0$ then COM tells us we are done. Next, COM, STR, and Axiom 6 tell us for $\beta \in (0, 1]$ and $e, j \in \mathcal{F}$, that $\beta x + (1-\beta)j \succeq \beta e + (1-\beta)j$ because for all $\gamma \in (0, \beta)$ we know $\beta x + (1-\beta)j \succ \gamma e + (\beta-\gamma)y + (1-\beta)j$ and the set $\{\alpha \in [0, 1] : \beta x + (1-\beta)j \succeq \alpha(\beta e + (1-\beta)j) + (1-\alpha)(\beta y + (1-\beta)j)\}$ is closed, and that $\beta e + (1-\beta)j \succeq \beta y + (1-\beta)j$, because for all $\gamma \in (0, \beta)$ we know $\gamma e + (\beta-\gamma)x + (1-\beta)j \succ \beta y + (1-\beta)j$ and the set $\{\alpha \in [0, 1] : \alpha(\beta e + (1-\beta)j) + (1-\alpha)(\beta x + (1-\beta)j) \succeq \beta y + (1-\beta)j\}$ is closed. Thus, letting f, g, and h, from the statement of Axiom 6 be defined $f = \beta x + (1-\beta)j$, $g = \beta y + (1-\beta)j$, and $h = \beta e + (1-\beta)j$, it must be that there is a $w \in [0, 100]$ such that $\beta e + (1-\beta)j \sim \beta w + (1-\beta)j$, and COM and STR tell us such w must be unique if $\beta > 0$. ■

Lemma 5. If the preferences of the DM satisfy WP, COM, WSTR, CON, TR, WCI, and Axiom 6, then if $\beta f + (1-\beta)g \succeq \beta w + (1-\beta)g$ for $\beta \in [0, 1]$, $w \in X$, and $f, g \in \mathcal{F}$, then for all $\alpha \in [0, \beta) : \alpha f + (1-\beta)g \succeq \alpha w + (1-\beta)g$, and if $\beta f + (1-\beta)g \succ \beta w + (1-\beta)g$ for $\beta \in (0, 1]$, $w \in X$, and $f, g \in \mathcal{F}$, then for all $\alpha \in (0, \beta) : \alpha f + (1-\beta)g \succ \alpha w + (1-\beta)g$.

Proof. Suppose $\beta f + (1-\beta)g \succeq \beta w + (1-\beta)g$ for $\beta \in [0, 1]$, $w \in X$, and $f, g \in \mathcal{F}$, then COM, CON, and WCI, tell us for all $\alpha \in [0, \beta)$:

$$\frac{\alpha}{\beta}(\beta f + (1-\beta)g) + \frac{\beta-\alpha}{\beta}(\beta w + (1-\beta)g) = \alpha f + (\beta-\alpha)w + (1-\beta)g$$

$$\succeq \beta w + (1-\beta)g = \alpha w + (\beta-\alpha)w + (1-\beta)g$$

$$\Rightarrow \alpha f + (1-\beta)g \succeq \alpha w + (1-\beta)g.$$

If, instead, $\beta f + (1-\beta)g \succ \beta w + (1-\beta)g$ for $\beta \in (0, 1]$, $w \in X$, and $f, g \in \mathcal{F}$, then Lemma 4 tells us $\exists \tilde{w} \in X$ such that $\beta f + (1-\beta)g \sim \beta\tilde{w} + (1-\beta)g$, and COM and STR (which is satisfied by Lemma 4) tell us $\tilde{w} > w$, then as we have just shown for all $\alpha \in (0, \beta) : \alpha f + (1-\beta)g \succeq \alpha\tilde{w} + (1-\beta)g$, and thus STR tells us

$\alpha f + (1-\beta)g \succ \alpha w + (1-\beta)g$. ■

Lemma 6. If the preferences of the DM satisfy WP, COM, WSTR, CON, TR, WCI, and Axiom 6, then if $\beta w + (1-\beta-\gamma)g \succeq \beta f + (1-\beta-\gamma)g$ for $\beta \in (0, 1)$, $\gamma \in [0, 1-\beta]$, $w \in X$, and $f, g \in \mathcal{F}$, then for all $\alpha \in [0, 1] :\ \alpha(\beta+\gamma)f + (1-\alpha)(\beta+\gamma)w + (1-\beta-\gamma)g \succeq (\beta+\gamma)f + (1-\beta-\gamma)g$, and if $\beta w + (1-\beta-\gamma)g \succ \beta f + (1-\beta-\gamma)g$ for $\beta \in (0, 1)$, $\gamma \in [0, 1-\beta]$, $w \in X$, and $f, g \in \mathcal{F}$, then for all $\alpha \in [0, 1) :\ \alpha(\beta+\gamma)f + (1-\alpha)(\beta+\gamma)w + (1-\beta-\gamma)g \succ (\beta+\gamma)f + (1-\beta-\gamma)g$.

Proof. If $\beta w + (1-\beta-\gamma)g \succeq \beta f + (1-\beta-\gamma)g$ for $\beta \in (0, 1)$, $\gamma \in [0, 1-\beta]$, $w \in X$, and $f, g \in \mathcal{F}$, then COM and Lemma 5 tell us $(\beta+\gamma)w + (1-\beta-\gamma)g \succeq (\beta+\gamma)f + (1-\beta-\gamma)g$, and COM and CON thus tell us for all $\alpha \in [0, 1] :\ \alpha(\beta+\gamma)f + (1-\alpha)(\beta+\gamma)w + (1-\beta-\gamma)g \succeq (\beta+\gamma)f + (1-\beta-\gamma)g$.

If $\beta w + (1-\beta-\gamma)g \succ \beta f + (1-\beta-\gamma)g$ for $\beta \in (0, 1)$, $\gamma \in [0, 1-\beta]$, $w \in X$, and $f, g \in \mathcal{F}$, then Lemma 4 tells us there is a $\tilde{w} \in X$ such that $\beta\tilde{w} + (1-\beta-\gamma)g \sim \beta f + (1-\beta-\gamma)g$, COM and STR (which is satisfied by Lemma 4) tell us $\tilde{w} < w$, and Lemma 5 tells us $(\beta+\gamma)\tilde{w} + (1-\beta-\gamma)g \succeq (\beta+\gamma)f + (1-\beta-\gamma)g$. Then COM and CON tell us that for all $\alpha \in [0, 1] :\ \alpha(\beta+\gamma)f + (1-\alpha)(\beta+\gamma)\tilde{w} + (1-\beta-\gamma)g \succeq (\beta+\gamma)f + (1-\beta-\gamma)g$, and thus STR tells us for $\alpha \in [0, 1) :\ \alpha(\beta+\gamma)f + (1-\alpha)(\beta+\gamma)w + (1-\beta-\gamma)g \succ (\beta+\gamma)f + (1-\beta-\gamma)g$. ■

Lemma 7. If the preferences of the DM satisfy WP, COM, WSTR, CON, TR, WCI, and Axiom 6, and they consider all three question in conjunction and assign equal and strictly positive weights to the probability equivalent questions, then they do not give an interval that is contained in the interior of their other interval.

Proof. Let $\alpha \in \{0, 0.01, 0.02, \ldots, 1\}$ be the chance that the DM selects to assign to betting on R as opposed to L in the randomization question. Let $g = \alpha R + (1-\alpha)L$. The specific composition of the act g does not impact our argument because all we need to do to show our desired result is focus on the answers to the probability equivalent questions. Assume that the DM gave an interval that is contained in the interior of their other interval and we shall reach a contradiction. Assume without loss of generality that $u_R > u_L$ and $l_R < l_L$ (this is without loss as the nature of g is irrelevant to the argument). Let $\beta \in [0, 1)$ denote the DM's weight on the randomization question when they answer the probability equivalent questions.

Since the DM does not want to decrease u_R COM, STR (which is satisfied by

Lemma 4), TR, and WCI tell us:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R-1)\Big)+\beta g \succ (1-\beta)(u_R-1)+\beta g,$$

because otherwise:

$$(1-\beta)(u_R-1)+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R-1)\Big)+\beta g \Rightarrow \forall x\in(u_R-1,\, u_R]:$$

$$(1-\beta)x+\beta g \succ (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(x)\Big)+\beta g,$$

and the DM could do strictly better by reducing u_R by 1 (doing so does not change the chosen act at u_R-1),

$$\text{so (using WCI) } \Rightarrow \frac{1-\beta}{4}R+\beta g \succ \frac{1-\beta}{4}(u_R-1)+\beta g.$$

Since the DM does not want to increase l_R COM, STR, TR, and WCI tell us:

$$(1-\beta)\Big(\frac{1}{4}L+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(l_R+1)\Big)+\beta g \succ (1-\beta)\frac{100}{2}+\beta g,$$

because otherwise:

$$(1-\beta)\frac{100}{2}+\beta g \succeq (1-\beta)\Big(\frac{1}{4}L+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(l_R+1)\Big)+\beta g \Rightarrow \forall x\in[l_R,\, l_R+1):$$

$$(1-\beta)\frac{100}{2}+\beta g \succ (1-\beta)\Big(\frac{1}{4}L+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(x)\Big)+\beta g,$$

and the DM could do strictly better by increasing l_R by 1 (doing so does not change the chosen act at l_R+1),

$$\text{so (using WCI) } \Rightarrow \frac{1-\beta}{4}L+\beta g \succ \frac{1-\beta}{4}(100-l_R-1)+\beta g.$$

Since the DM neither benefits from switching their upper bounds with each other or switching their lower bounds with each other, which would replace a $\frac{1-\beta}{4}R$ with a $\frac{1-\beta}{4}L$ or replace a $\frac{1-\beta}{4}L$ with a $\frac{1-\beta}{4}R$ respectively, COM tells us (using CON and TR):

$$\frac{1-\beta}{4}R+\beta g \sim \frac{1-\beta}{4}L+\beta g,$$

$$\Rightarrow \frac{1-\beta}{4}\Big(\frac{1}{2}R+\frac{1}{2}L\Big)+\beta g=\frac{1-\beta}{4}\frac{100}{2}+\beta g\succ\frac{1-\beta}{4}(u_R-1)+\beta g$$

$$\text{and } \frac{1-\beta}{4}\Big(\frac{1}{2}R+\frac{1}{2}L\Big)+\beta g=\frac{1-\beta}{4}\frac{100}{2}+\beta g\succ\frac{1-\beta}{4}(100-l_R-1)+\beta g.$$

So, if $u_R \geq 51$ we have a contradiction with COM and STR, and if $u_R \leq 50$ then $l_R < 50$ and we again have a contradiction with COM and STR. ■

Lemma 7 is interesting because it means that there are certain responses to the probability equivalent questions that are admissible if the DM is not believed to be integrating, but can be rejected if they are integrating. The same is true for Lemma 8.

Lemma 8. If the preferences of the DM satisfy WP, COM, WSTR, CON, TR, WCI, and Axiom 6, and they answer all three questions in conjunction and assign equal and strictly positive weights to the probability equivalent questions, then they would never give upper bounds with $\max(u_R, u_L) < 50$, or lower bounds $l_R = l_L < 50$ unless $l_R = l_L = \min(u_L, u_R)$, and if $\max(u_L, u_R) = 50$ then $\min(l_R, l_L) \geq 49$.

Proof. Let $\alpha \in \{0, 0.01, 0.02, \ldots, 1\}$ be the chance that the DM selects to assign to betting on R as opposed to L in the randomization question. Let $g = \alpha R + (1-\alpha)L$. The specific composition of the act g does not impact our argument because all we need to do to show our desired result is focus on the answers to the probability equivalent questions. Let $\beta \in [0, 1)$ denote the DM's weight on the randomization question when they answer the probability equivalent questions. Assume without loss of generality that $u_R \geq u_L$ (this is without loss as the nature of g is irrelevant to the argument).

If $u_R = u_L < 49$, COM and STR (which is satisfied by Lemma 4) tell us the DM could strictly benefit from increasing their upper bounds. If $l_R = l_L < \min(u_L, 50)$ COM and STR tell us the DM could strictly benefit from increasing their lower bounds. If $u_R = u_L = 49$ then if $l_R = l_L = 49$ then COM and STR tell us the DM could strictly benefit from increasing their lower and upper bounds to 50. If $u_R = u_L = 49$ and $\min(l_L, l_R) < \max(l_L, l_R)$, in which case we can assume without loss of generality that $\min(l_L, l_R) = l_R$, and thus COM, STR, and WCI tell us that since the DM does not increase l_R:

$$(1-\beta)\Big(\frac{1}{4}L+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(l_R+1)\Big)+\beta g\succ(1-\beta)\frac{100}{2}+\beta g,$$

but then the DM could do strictly better by increasing u_L by one since $l_R + 1 \leq 49$. If $50 > u_R > u_L$ then Lemma 7 tells us $l_R \geq l_L$ and COM, STR, and WCI tell us that since the DM does not increase u_R to 50 (using WCI and Lemma 6):

$$(1-\beta)\frac{100}{2} + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}\frac{100}{2}\Big) + \beta g$$

$$\Rightarrow (1-\beta)\frac{1}{4}\frac{100}{2} + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g$$

$$\Rightarrow (1-\beta)\frac{1}{2}\frac{100}{2} + \beta g \succeq (1-\beta)\frac{1}{2}R + \beta g,$$

and COM and STR tell us the DM could do strictly better by increasing all bounds to 50.

We have thus shown $\max(u_L, u_R) = u_R \geq 50$. For the rest of the proof assume $u_R = 50$. We know from COM, STR, and WCI, that since the DM does not increase u_R:

$$(1-\beta)51 + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}51\Big) + \beta g$$

$$\Rightarrow (1-\beta)\frac{1}{4}51 + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g.$$

COM, STR, and Lemma 4 thus tell us there is $w \leq 51$ such that $(1-\beta)\frac{1}{4}R + \beta g \sim (1-\beta)\frac{1}{4}w + \beta g$. We next show $\min(l_L, l_R) \geq 49$. If $\min(l_L, l_R) < 49$ it must be $l_L < 49$, because if not then $l_R < \min(l_L, 49)$, and Lemma 7 tells us $u_L = 50$, and since the DM does not increase l_R COM, STR, and WCI, tell us:

$$(1-\beta)\Big(\frac{1}{4}L + \frac{1}{2}50 + \frac{1}{4}(l_R+1)\Big) + \beta g \succ (1-\beta)50 + \beta g,$$

and thus the DM would strictly benefit from increasing u_L to 51, so if $\min(l_L, l_R) < 49$ it must be $l_L < 49$. If $l_L < 49$, $u_L = l_L$, and $u_R > l_R$, then if $l_R = u_L$ the DM would strictly benefit from increasing l_R, u_L, and l_L, to 49, so if $l_L < 49$, $u_L = l_L$, and $u_R > l_R$, then $l_R > u_L$ (using Lemma 7) and since the DM does not increase l_R COM, STR, and WCI, tell us:

$$(1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}\frac{100}{2}\Big) + \beta g \succ (1-\beta)\Big(\frac{1}{2}R + \frac{1}{2}\frac{100}{2}\Big) + \beta g,$$

and the DM could thus do strictly better by increasing u_L. If $l_L < 49$ and $l_L < u_L$, then since the DM does not benefit from increasing l_L, COM, STR, and WCI, tell us (using TR):

$$(1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(l_L+1)\Big)+\beta g \succ (1-\beta)\frac{100}{2}+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}51+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(l_L+1)\Big)+\beta g \succ (1-\beta)\frac{100}{2}+\beta g$$

$$\Rightarrow l_L \geq 49,$$

and we have a contradiction. If $l_L < 49$, $l_R = u_R$, and $u_L = l_L$, then since the DM does not benefit from increasing u_L COM, STR, and WCI, tell us (using TR and WCI):

$$(1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(u_L+1)\Big)+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(u_L+1)\Big)+\beta g.$$

$$\Rightarrow (1-\beta)\frac{1}{2}R+\beta g \succeq (1-\beta)\Big(\frac{1}{4}w+\frac{100-(u_L+1)}{4}\Big)+\beta g,$$

but then Lemma 5, COM, and STR, tell us $w = 51$ and the DM could do strictly better by increasing u_R and l_R to 51. ■

Lemma 9. If the preferences of the DM satisfy WP, COM, WSTR, CON, TR, WCI, and Axiom 6, and they answer all three questions in conjunction and assign equal and strictly positive weights to the probability equivalent questions, then if $\max(u_L, u_R) > 50$ and $\min(u_L, u_R) \leq 50$, then $\max(l_L, l_R) \geq \min(u_L, u_R)$, $\min(l_L, l_R)+\max(u_L, u_R) \in [99, 101]$, and either $\min(u_L, u_R) \in [\max(l_L, l_R)-1, \max(l_L, l_R)]$ or $\max(l_L, l_R) + \min(u_L, u_R) \in [99, 101]$.

Proof. Let $\alpha \in \{0, 0.01, 0.02, \ldots, 1\}$ be the chance that the DM selects to assign to betting on R as opposed to L in the randomization question. Let $g = \alpha R + (1-\alpha)L$. The specific composition of the act g does not impact our argument because all we need to do to show our desired result is focus on the answers to the probability equivalent questions. Let $\beta \in [0, 1)$ denote the DM's weight on the randomization question when they answer the probability equivalent questions. Assume without loss of generality that $u_R \geq u_L$ (this is without loss as the nature of g is irrelevant to the argument), and then assume $u_R > 50$ and $u_L \leq 50$. Lemma 7 tells us $l_R \geq l_L$.

First, $\max(l_L, l_R) = l_R \geq \min(u_L, u_R) = u_L$, because otherwise COM, STR (which is satisfied by Lemma 4), WCI, and the fact that the DM does not lower u_R

tells us:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}\frac{100}{2}\Big)+\beta g \succ (1-\beta)\frac{100}{2}+\beta g$$
$$\Rightarrow (1-\beta)\frac{1}{4}R+\beta g \succ (1-\beta)\frac{1}{4}\frac{100}{2}+\beta g,$$

and the DM could do strictly better by increasing l_R to u_L (which locally changes the chosen act from $(1-\beta)(\frac{1}{2}(\frac{1}{2}R+\frac{1}{2}L)+\frac{1}{2}y)+\beta g=(1-\beta)(\frac{1}{2}\frac{100}{2}+\frac{1}{2}y)+\beta g$ to $(1-\beta)(\frac{1}{4}R+\frac{1}{2}(\frac{1}{2}R+\frac{1}{2}L)+\frac{1}{4}y)+\beta g=(1-\beta)(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}y)+\beta g)$.

If $u_R>l_R$ then COM, STR, WCI, and the fact that the DM does not lower u_R tells us:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R-1)\Big)+\beta g \succ (1-\beta)(u_R-1)+\beta g$$
$$\Rightarrow (1-\beta)\frac{1}{4}R+\beta g \succ (1-\beta)\frac{1}{4}(u_R-1)+\beta g,$$

but if $u_R=l_R$ then COM, STR, TR, WCI, Lemma 6, and the fact that the DM does not lower u_R and l_R (which locally changes the chosen act, depending on if the act is equal to the original u_R or the new u_R which is more than u_L or the new u_R which is equal to $u_L=50$ or between the original and new u_R, from $(1-\beta)(\frac{1}{4}R+\frac{3}{4}u_R)+\beta g$ to $(1-\beta)u_R+\beta g$ or from $(1-\beta)(\frac{1}{2}R+\frac{1}{2}(u_R-1))+\beta g$ to $(1-\beta)(\frac{1}{4}R+\frac{3}{4}(u_R-1))+\beta g$ or from $(1-\beta)(\frac{1}{4}R+\frac{3}{4}(50))+\beta g$ to $(1-\beta)50+\beta g$ or from $(1-\beta)(\frac{1}{2}R+\frac{1}{2}y)+\beta g$ to $(1-\beta)y+\beta g$) tells us the same thing:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R-1)\Big)+\beta g \succ (1-\beta)(u_R-1)+\beta g$$

$$\Rightarrow (1-\beta)\frac{1}{4}R+\beta g \succ (1-\beta)\frac{1}{4}(u_R-1)+\beta g,$$

because if:

$$(1-\beta)(u_R-1)+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R-1)\Big)+\beta g$$

then WCI and Lemma 6 tell us:

$$(1-\beta)\frac{1}{2}(u_R-1)+\beta g \succeq (1-\beta)\frac{1}{2}R+\beta g \text{ and } (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(u_R-1)\Big)+\beta g \succeq (1-\beta)\frac{1}{2}R+\beta g.$$

If $u_L>l_L$ then COM, STR, WCI, and the fact that the DM does not increase

l_L tells us:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(l_L+1)\Big)+\beta g \succ (1-\beta)\frac{100}{2}+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_L+1)\Big)+\beta g \succ (1-\beta)\Big(\frac{100}{4}\Big)+\beta g,$$

but if $u_L = l_L < l_R$ then the fact that the DM does not increase u_L and l_L (which locally changes the chosen act, depending on if the act is equal to the original u_L or the new u_L which is less than l_R or the new u_L which is equal to l_R or between the original and new u_L, from $(1-\beta)(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}l_L)+\beta g$ to $(1-\beta)\frac{100}{2}+\beta g$ or from $(1-\beta)(\frac{1}{2}R+\frac{1}{2}(l_L+1))+\beta g$ to $(1-\beta)(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(l_L+1))+\beta g$ or from $(1-\beta)(\frac{1}{4}R+\frac{3}{4}(l_L+1))+\beta g$ to $(1-\beta)\Big(\frac{1}{2}\frac{100}{2}+\frac{1}{2}(l_L+1)\Big)+\beta g$ or from $(1-\beta)(\frac{1}{2}R+\frac{1}{2}y)+\beta g$ to $(1-\beta)\frac{100}{2}+\beta g$), COM, STR, TR, WCI, and Lemma 6, tell us the same thing:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(l_L+1)\Big)+\beta g \succ (1-\beta)\frac{100}{2}+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_L+1)\Big)+\beta g \succ (1-\beta)\Big(\frac{100}{4}\Big)+\beta g,$$

because if (using WCI):

$$(1-\beta)\Big(\frac{100}{4}\Big)+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_L+1)\Big)+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{100-(l_L+1)}{4}\Big)+\beta g \succeq (1-\beta)\frac{1}{4}R+\beta g$$

then WCI and Lemma 6 tell us:

$$(1-\beta)\frac{100-(l_L+1)}{2}+\beta g \succeq (1-\beta)\frac{1}{2}R+\beta g$$

$$\text{and } (1-\beta)\Big(\frac{1}{4}R+\frac{100-(l_L+1)}{4}\Big)+\beta g \succeq (1-\beta)\frac{1}{2}R+\beta g,$$

while if $u_L = l_L = l_R$, then either $u_L = l_L = l_R \geq 49$ and $u_R = 51$, in which case we are done, or, COM, STR, and WCI, tell us we can again reach the same conclusion because otherwise, if:

$$(1-\beta)\Big(\frac{100}{4}\Big)+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_L+1)\Big)+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{100-(l_L+1)}{4}\Big) + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{100-(l_L+1)}{2}\Big) + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{100-(l_L+1)}{4}\Big) + \beta g,$$

then either $u_L = l_L = l_R < 49$ and thus for $y < l_L + 1 \leq 49$ (using STR and WCI):

$$(1-\beta)\Big(\frac{100-y}{2}\Big) + \beta g \succ (1-\beta)\Big(\frac{1}{4}R + \frac{y}{4}\Big) + \beta g$$

$$\Rightarrow (1-\beta)\frac{100}{2} + \beta g \succ (1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}y\Big) + \beta g$$

and so the DM can strictly benefit from increasing u_L, l_L, and l_R, by one (which locally changes the chosen act, depending on if the act is equal to the original l_L or the new l_L or between them, from $(1-\beta)(\frac{1}{2}\frac{100}{2} + \frac{1}{2}l_L) + \beta g$ to $(1-\beta)\frac{100}{2} + \beta g$ or from $(1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(l_L+1)\Big) + \beta g$ to $(1-\beta)\Big(\frac{1}{2}\frac{100}{2} + \frac{1}{2}(l_L+1)\Big) + \beta g$ or from $(1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}y\Big) + \beta g$ to $(1-\beta)\frac{100}{2} + \beta g$), or $u_L = l_L = l_R \geq 49$ and $u_R > 51$ in which case for $y > u_R - 1 \geq 51$ (using STR and WCI):

$$(1-\beta)\Big(\frac{100-(l_L+1)}{4}\Big) + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g$$

$$\Rightarrow (1-\beta)y + \beta g \succ (1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}y\Big) + \beta g,$$

and so the DM can strictly benefit from decreasing u_R by one (which locally changes the chosen act from $(1-\beta)(\frac{1}{4}R + \frac{3}{4}y) + \beta g$ to $(1-\beta)y + \beta g$). So, we know:

$$(1-\beta)\frac{1}{4}R + \beta g \succ (1-\beta)\frac{1}{4}(u_R - 1) + \beta g \text{ (from the previous paragraph)}$$

$$\text{and } (1-\beta)\Big(\frac{1}{4}R + \frac{1}{4}(l_L+1)\Big) + \beta g \succ (1-\beta)\Big(\frac{100}{4}\Big) + \beta g.$$

If $u_R = 100$ then either $l_L = 0$ (so $l_L + u_R = 100$), or $l_L > 0$ and COM, STR, WCI, and the fact that the DM does not lower l_L tells us (using COM, STR, TR, and WCI):

$$(1-\beta)\frac{100}{2} + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}(l_L - 1)\Big) + \beta g,$$

and because we showed:

$$(1-\beta)\frac{1}{4}R + \beta g \succ (1-\beta)\frac{1}{4}(u_R - 1) + \beta g,$$

we then know:

$$(1-\beta)\frac{100}{4} + \beta g \succ (1-\beta)\Big(\frac{1}{4}(u_R - 1) + \frac{1}{4}(l_L - 1)\Big) + \beta g$$

$$\Rightarrow l_L + u_R = 101,$$

and if $u_R < 100$ then COM, STR, WCI, and the fact that the DM does not increase u_R tells us (using COM, STR, TR, and WCI):

$$(1-\beta)(u_R + 1) + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(u_R + 1)\Big) + \beta g$$

$$\Rightarrow (1-\beta)\frac{1}{4}(u_R + 1) + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g$$

and because we showed:

$$(1-\beta)\Big(\frac{1}{4}R + \frac{1}{4}(l_L + 1)\Big) + \beta g \succ (1-\beta)\Big(\frac{100}{4}\Big) + \beta g,$$

we then know:

$$(1-\beta)\frac{1}{4}(u_R + 1) + \beta g \succ (1-\beta)\frac{1}{4}(100 - (l_L + 1)) + \beta g$$

$$\Rightarrow u_R + l_L > 98,$$

and either $l_L = 0$ (so $l_L + u_R = 99$), or $l_L > 0$ and COM, STR, WCI, and the fact that the DM does not lower l_L tells us (using COM, STR, TR, and WCI):

$$(1-\beta)\frac{100}{2} + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}(l_L - 1)\Big) + \beta g$$

and because we showed:

$$(1-\beta)\frac{1}{4}R + \beta g \succ (1-\beta)\frac{1}{4}(u_R - 1) + \beta g,$$

we then know:

$$(1-\beta)\frac{100}{4}+\beta g \succ (1-\beta)\Big(\frac{1}{4}(u_R-1)+\frac{1}{4}(l_L-1)\Big)+\beta g$$

$$\Rightarrow l_L+u_R<102,$$

so $l_L+u_R\in[99,\,101]$.

For the rest of the proof assume $l_R>u_L+1$. If $l_R<u_R$ then the fact that the DM does not increase l_R, COM, STR, and WCI, tell us:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(l_R+1)\Big)+\beta g \succ (1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(l_R+1)\Big)+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_R+1)\Big)+\beta g \succ (1-\beta)\frac{1}{2}R+\beta g,$$

but if $l_R=u_R$ then either $u_R=100$ (in which case the same conclusion can be reached with COM and STR) or COM, STR, TR, WCI, and Lemma 6 tell us, since:

$$(1-\beta)\frac{1}{2}R+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_R+1)\Big)+\beta g,$$

implies that (for $y<l_R+1$):

$$(1-\beta)\frac{1}{4}R+\beta g \succeq (1-\beta)\Big(\frac{1}{4}(l_R+1)\Big)+\beta g \text{ and } (1-\beta)\frac{1}{2}R+\beta g \succ (1-\beta)\frac{1}{2}y+\beta g,$$

and the fact that the DM does not increase l_R and u_R (which locally changes the chosen act, depending on if the act is equal to the original l_R or the new l_R or between them, from $(1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(l_R)\Big)+\beta g$ to $(1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(l_R)\Big)+\beta g$ or from $(1-\beta)(l_R+1)+\beta g$ to $(1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(l_R+1)\Big)+\beta g$ or from $(1-\beta)y+\beta g$ to $(1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}y\Big)+\beta g$) tells us the same thing:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(l_R+1)\Big)+\beta g \succ (1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(l_R+1)\Big)+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_R+1)\Big)+\beta g \succ (1-\beta)\frac{1}{2}R+\beta g,$$

and either way the fact that the DM does not decrease l_R, COM, STR, and WCI, tell us

$$(1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(l_R-1)\Big)+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(l_R-1)\Big)+\beta g$$
$$\Rightarrow (1-\beta)\frac{1}{2}R+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_R-1)\Big)+\beta g.$$

If $u_L > l_L$ then the fact that the DM does not decrease u_L, COM, STR, and WCI, tell us:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(u_L-1)\Big)+\beta g \succ (1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(u_L-1)\Big)+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(100-(u_L-1))\Big)+\beta g \succ (1-\beta)\frac{1}{2}R+\beta g,$$

but if $u_L = l_L$ then either $u_L = 0$ (in which case the same conclusion can be reached with COM and STR) or COM, STR, TR, WCI, and Lemma 6 tell us, since:

$$(1-\beta)\frac{1}{2}R+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(100-(u_L-1))\Big)+\beta g$$

implies that (for $y > u_L - 1$):

$$(1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(u_L-1)\Big)+\beta g \succeq (1-\beta)\frac{100}{2}+\beta g,$$

$$\text{and } (1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}y\Big)+\beta g \succ (1-\beta)\frac{100}{2}+\beta g$$

and the fact that the DM does not decrease u_L and l_L (which locally changes the chosen act, depending on if the act is equal to the original u_L or the new u_L or between them, from $(1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(u_L)\Big)+\beta g$ to $(1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(u_L)\Big)+\beta g$ or from $(1-\beta)\frac{100}{2}+\beta g$ to $(1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(u_L-1)\Big)+\beta g$ or from $(1-\beta)\frac{100}{2}+\beta g$ to $(1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}y\Big)+\beta g$) tells us the same thing:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(u_L-1)\Big)+\beta g \succ (1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(u_L-1)\Big)+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(100-(u_L-1))\Big)+\beta g \succ (1-\beta)\frac{1}{2}R+\beta g,$$

and either way the fact that the DM does not increase u_L, COM, STR, and WCI, tell us:

$$(1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(u_L+1)\Big)+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(u_L+1)\Big)+\beta g$$

$$\Rightarrow (1-\beta)\frac{1}{2}R+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(100-(u_L+1))\Big)+\beta g.$$

Thus, because we showed:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_R+1)\Big)+\beta g \succ (1-\beta)\frac{1}{2}R+\beta g,$$

$$(1-\beta)\frac{1}{2}R+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_R-1)\Big)+\beta g,$$

$$(1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(100-(u_L-1))\Big)+\beta g \succ (1-\beta)\frac{1}{2}R+\beta g,$$

$$\text{and } (1-\beta)\frac{1}{2}R+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(100-(u_L+1))\Big)+\beta g,$$

COM, STR, and TR, tell us $l_R+u_L \in [99, 101]$. ■

Theorem 3.1. If the DM answers all three questions in conjunction and assigns equal and strictly positive weights to the probability equivalent questions, and their preferences satisfy WP, COM, WSTR, CON, TR, WCI, and Axiom 6, then they do not give an interval that is contained in the interior of their other interval, and: $\max(u_R, u_L)+\min(l_L, l_R) \in [99, 101]$, $\min(u_R, u_L) \leq \max(l_R, l_L)$, and either $\min(u_R, u_L) \in [\max(l_L, l_R)-1, \max(l_L, l_R)]$ or $\min(u_R, u_L)+\max(l_R, l_L) \in [99, 101]$.

Proof. Let $\alpha \in \{0, 0.01, 0.02, \ldots, 1\}$ be the chance that the DM selects to assign to betting on R as opposed to L in the randomization question. Let $g=\alpha R+(1-\alpha)L$. The specific composition of the act g does not impact our argument because all we need to do to show our desired result is focus on the answers to the probability equivalent questions. Let $\beta \in [0, 1)$ denote the DM's weight on the randomization question when they answer the probability equivalent questions. If $u_L+u_R \leq 101$ then we are done by Lemma 7, Lemma 8, and Lemma 9. Assume for the rest of the proof that $u_L+u_R \geq 102$, and assume without loss of generality that $u_R \geq u_L$ (this is without loss as the nature of g is irrelevant to the argument, and thus $u_R \geq 51$ and $u_L \geq 2$). Next we will address all of the potential cases for our bounds.

Case 1: If, $u_R > u_L > l_R$ then, the L interval is non-degenerate (in addition to the R interval being non-degenerate given the conditions of this case) because Lemma 7 tells us $l_R \geq l_L$. Given COM, STR (which is satisfied by Lemma 4), and WCI, the upper bounds thus imply (since the DM would not strictly benefit from lowering an upper bound):

$$(1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(u_R - 1)\Big) + \beta g \succ (1-\beta)(u_R - 1) + \beta g,$$

$$\text{and } (1-\beta)\Big(\frac{1}{2}\frac{100}{2} + \frac{1}{2}(u_L - 1)\Big) + \beta g \succ (1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(u_L - 1)\Big) + \beta g.$$

Thus, WCI tells us:

$$(1-\beta)\Big(\frac{1}{4}R + \frac{3}{8}(u_R - 1) + \frac{3}{8}(u_L - 1)\Big) + \beta g$$

$$\succ (1-\beta)\Big(\frac{3}{8}(u_R - 1) + \frac{1}{8}(u_L - 1) + \frac{4}{8}\frac{(u_R - 1) + (u_L - 1)}{2}\Big) + \beta g$$

$$\text{and } (1-\beta)\Big(\frac{100}{4} + \frac{3}{8}(u_R - 1) + \frac{1}{8}(u_L - 1)\Big) + \beta g$$

$$\succ (1-\beta)\Big(\frac{1}{4}R + \frac{3}{8}(u_R - 1) + \frac{3}{8}(u_L - 1)\Big) + \beta g.$$

Putting these together using TR:

$$(1-\beta)\Big(\frac{100}{4} + \frac{3}{8}(u_R - 1) + \frac{1}{8}(u_L - 1)\Big) + \beta g$$

$$\succ (1-\beta)\Big(\frac{3}{8}(u_R - 1) + \frac{1}{8}(u_L - 1) + \frac{4}{8}\frac{(u_R - 1) + (u_L - 1)}{2}\Big) + \beta g,$$

which contradicts COM, STR, and TR. This all means that Case 1 is not possible.

Case 2: If, instead, $u_R > u_L = l_R$ then COM, STR, WCI, and the fact that the DM does not change u_R tells us (using WCI):

$$(1-\beta)(u_R + 1) + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(u_R + 1)\Big) + \beta g,$$

and

$$(1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(u_R - 1)\Big) + \beta g \succ (1-\beta)(u_R - 1) + \beta g$$

$$\Rightarrow (1-\beta)\frac{1}{4}(u_R + 1) + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g \succ (1-\beta)\frac{1}{4}(u_R - 1) + \beta g.$$

(This argument works unless $u_R = 100$, in which case COM, STR, and Axiom 6, tell us $(1-\beta)\frac{1}{4}u_R + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g \succ (1-\beta)\frac{1}{4}(u_R - 1) + \beta g$, and the rest of the argument becomes easier). COM, STR, WCI, and the fact that the DM does not decrease l_L tells us (using TR and WCI):

$$(1-\beta)\frac{100}{2} + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}(l_L - 1)\Big) + \beta g$$

$$\Rightarrow (1-\beta)\frac{1}{2}\frac{100}{2} + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{1}{4}(l_L - 1)\Big) + \beta g$$

$$\Rightarrow (1-\beta)\frac{100}{4} + \beta g \succ (1-\beta)\frac{1}{4}(u_R - 1 + l_L - 1) + \beta g.$$

(This argument works unless $l_L = 0$, in which case it must be that $l_R = u_L > l_L$ because $u_L + u_R \geq 102$, and then COM, STR, WCI, and the fact that the DM does not increase l_L tells us $(1-\beta)(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}) + \beta g \succ (1-\beta)50 + \beta g$, and then COM, STR, TR, and WCI, tell us $u_R \geq 99$ and we are done.) Thus COM and STR tell us $l_L + u_R < 102$, and since $u_R + u_L \geq 102$, we know $l_L < u_L$, and we can conclude from COM, STR, WCI, and the DM's choice of l_L that (using COM, STR, TR, and WCI):

$$(1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}(l_L + 1)\Big) + \beta g \succ (1-\beta)\frac{100}{2} + \beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}(u_R + 1) + \frac{1}{4}(l_L + 1)\Big) + \beta g \succ (1-\beta)\frac{50}{2} + \beta g$$

$$\Rightarrow u_R + l_L > 98.$$

Together, this all means that if we are in Case 2 then $u_R + l_L \in [99, 101]$.

Case 3: Suppose, instead, $u_R \geq l_R = u_L + 1$. Since the DM does not want to increase u_R COM, STR, and WCI, tell us (using WCI):

$$(1-\beta)(u_R + 1) + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(u_R + 1)\Big) + \beta g$$

$$\Rightarrow (1-\beta)\frac{1}{4}(u_R + 1) + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g. \tag{1}$$

(This argument works unless $u_R = 100$, in which case COM, STR, and Axiom 6, tell us $(1-\beta)\frac{1}{4}u_R + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g$ and the rest of the argument is easier.) Since the DM does not lower l_R, COM, STR, and WCI, tell us (using COM, CON, and WCI) either the DM prefers the act they chose at u_L to the one they would get instead if they lowered l_R by one:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}u_L\Big)+\beta g \succeq (1-\beta)\Big(\frac{1}{2}\frac{100}{2}+\frac{1}{2}u_L\Big)+\beta g$$

$$\Rightarrow (1-\beta)\frac{1}{4}R+\beta g \succeq (1-\beta)\frac{1}{4}u_L+\beta g,$$

or the DM strictly prefers one of the acts between u_L and l_R to the one they would get instead if they lowered l_R by one:

$$(1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}u_L\Big)+\beta g \succ (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}u_L\Big)+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}u_L\Big)+\beta g \succeq (1-\beta)u_L+\beta g$$

$$\Rightarrow (1-\beta)\frac{1}{4}R+\beta g \succeq (1-\beta)\frac{1}{4}u_L+\beta g.$$

Further, COM and Lemma 5 tell us:

$$\text{if } (1-\beta)\frac{1}{4}R+\beta g \sim (1-\beta)\frac{1}{4}u_L+\beta g \text{ then } (1-\beta)\frac{1}{2}u_L+\beta g \succeq (1-\beta)\frac{1}{2}R+\beta g,$$

and the DM could do strictly better by lowering both u_R and l_R to u_L by COM, STR, and WCI, so:

$$(1-\beta)\frac{1}{4}R+\beta g \succ (1-\beta)\frac{1}{4}u_L+\beta g. \tag{2}$$

If $u_R = l_R = u_L + 1$ it must then be that $l_L < u_L$ since $u_L > 50$ and so otherwise the DM could do strictly better by lowering l_L to 50, and since the DM does not want to increase l_L COM tells us (using COM, STR, TR, and WCI, and equation (1) above):

$$(1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(l_L+1)\Big)+\beta g \succ (1-\beta)\frac{100}{2}+\beta g$$

$$\Rightarrow u_R + l_L \geq 99.$$

If $u_R = l_R$, then if $l_L = 0$ we have $u_R + l_L \leq 101$, while if $l_L > 0$, since the DM does not want to decrease l_L, COM, STR, and WCI, tell us (using equation (2) above, COM, STR, TR, and WCI):

$$(1-\beta)\frac{100}{2}+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(l_L-1)\Big)+\beta g$$

$$\Rightarrow l_L + u_R \leq 101.$$

If, instead $u_R > l_R = u_L + 1$, then since the DM does not want to decrease u_R COM, STR, and WCI tell us:

$$(1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(u_R - 1)\Big) + \beta g \succ (1-\beta)(u_R - 1) + \beta g$$

$$\Rightarrow (1-\beta)\frac{1}{4}R + \beta g \succ (1-\beta)\frac{1}{4}(u_R - 1) + \beta g,$$

and either $l_L = 0$ so $u_R + l_L \leq 100$, or $l_L > 0$ and then since the DM does not want to decrease l_L COM, STR, and WCI, tell us (using COM, STR, TR, and WCI):

$$(1-\beta)\frac{100}{2} + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}(l_L - 1)\Big) + \beta g$$

$$\Rightarrow l_L + u_R \leq 101.$$

Thus, if $u_R > l_R = u_L + 1$, $l_L < u_L$ since $u_R + u_L \geq 102$ and $u_R + l_L \leq 101$, and since the DM does not benefit from increasing l_L, COM, STR, and WCI, tell us (using COM, STR, TR, WCI, and equation (1) above):

$$(1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}(l_L + 1)\Big) + \beta g \succ (1-\beta)\frac{100}{2} + \beta g$$

$$\Rightarrow u_R + l_L \geq 99.$$

Case 4: Suppose, instead, $u_R \geq l_R > u_L + 1$. If $u_R = l_R$ then, since the DM does not want to increase u_R or decrease l_R, COM, STR, and WCI tell us:

$$(1-\beta)(u_R + 1) + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(u_R + 1)\Big) + \beta g,$$

$$\text{and } (1-\beta)\Big(\frac{1}{2}R + \frac{1}{2}(u_R - 1)\Big) + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(u_R - 1)\Big) + \beta g,$$

further, COM, WCI, and Lemma 6 tell us:

$$(1-\beta)(u_R + 1) + \beta g \succ (1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(u_R + 1)\Big) + \beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(u_R + 1)\Big) + \beta g \succ (1-\beta)\Big(\frac{1}{2}R + \frac{1}{2}(u_R + 1)\Big) + \beta g,$$

$$\text{and } (1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(u_R-1)\Big)+\beta g \succ (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R-1)\Big)+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R-1)\Big)+\beta g \succ (1-\beta)(u_R-1)+\beta g,$$

while COM, STR, TR, WCI, and the fact that the DM does not benefit from increasing or decreasing both u_R and l_R tells us:

$$(1-\beta)(u_R+1)+\beta g \sim (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R+1)\Big)+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R+1)\Big)+\beta g \succ (1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(u_R+1)\Big)+\beta g,$$

because if:

$$(1-\beta)(u_R+1)+\beta g \sim (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R+1)\Big)+\beta g$$

$$\text{and } (1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(u_R+1)\Big)+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R+1)\Big)+\beta g,$$

then:

$$(1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}u_R\Big)+\beta g \succ (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}u_R\Big)+\beta g$$

$$\text{and for } x\in(u_R,\, u_R+1) : (1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}x\Big)+\beta g \succ (1-\beta)x+\beta g$$

so the DM could strictly benefit from increasing both u_R and l_R by 1, and further:

$$(1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(u_R-1)\Big)+\beta g \sim (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R-1)\Big)+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R-1)\Big)+\beta g \succ (1-\beta)(u_R-1)+\beta g,$$

because if:

$$(1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(u_R-1)\Big)+\beta g \sim (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R-1)\Big)+\beta g$$

$$\text{and } (1-\beta)(u_R-1)+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R-1)\Big)+\beta g,$$

then:

$$(1-\beta)u_R+\beta g \succ (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}u_R\Big)+\beta g$$

$$\text{and for } x \in (u_R - 1,\, u_R) : (1-\beta)x + \beta g \succ (1-\beta)\Big(\frac{1}{2}R + \frac{1}{2}x\Big) + \beta g$$

so the DM could strictly benefit from decreasing both u_R and l_R by one, so, either way, using TR and WCI:

$$(1-\beta)\frac{1}{4}(u_R+1) + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g \succ (1-\beta)\frac{1}{4}(u_R-1) + \beta g,$$

$$\text{and } (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_R+1)\Big)+\beta g \succ (1-\beta)\frac{1}{2}R+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_R-1)\Big)+\beta g.$$

(This argument works unless $u_R = 100$, in which case we can argue in a similar fashion that $(1-\beta)\frac{1}{4}u_R+\beta g \succeq (1-\beta)\frac{1}{4}R+\beta g \succ (1-\beta)\frac{1}{4}(u_R-1)+\beta g$, $(1-\beta)(\frac{1}{4}R+\frac{1}{4}l_R)+\beta g \succ (1-\beta)\frac{1}{2}R+\beta g \succeq (1-\beta)(\frac{1}{4}R+\frac{1}{4}(l_R-1))+\beta g$, and the rest of the argument becomes easier, or $(1-\beta)(\frac{1}{4}R+\frac{1}{4}l_R)+\beta g \sim (1-\beta)\frac{1}{2}R+\beta g$, in which case COM, STR, TR, Lemma 4, and Lemma 5, tell us $(1-\beta)\frac{1}{2}R+\beta g \sim (1-\beta)\frac{1}{2}100+\beta g$ because if $(1-\beta)\frac{1}{2}100+\beta g \succ (1-\beta)\frac{1}{2}R+\beta g$ then $(1-\beta)\frac{1}{2}R+\beta g \sim (1-\beta)\frac{1}{2}x+\beta g$ with $x < 100$ and we have $(1-\beta)\frac{1}{2}x+\beta g \sim (1-\beta)\frac{1}{4}R+\frac{1}{4}100+\beta g \succeq (1-\beta)\frac{1}{4}x+\frac{1}{4}100+\beta g$ which violates COM and STR while if $(1-\beta)\frac{1}{2}R+\beta g \succ (1-\beta)\frac{1}{2}100+\beta g$ then $(1-\beta)\frac{1}{2}R+\beta g \sim (1-\beta)\frac{1}{2}x+\beta g$ with $x > 100$ which violates COM and STR, and thus WCI and TR tell us $(1-\beta)\frac{1}{4}R+\beta g \sim (1-\beta)\frac{1}{4}100+\beta g$, and so $u_L \leq 1$ because otherwise the DM could strictly benefit from lowering u_L and or l_L, and we are done.)

If, instead, $u_R > l_R$ then we can reach the same conclusions (summarized in equations (3) and (4) below) because the upper and lower bounds for R, COM, STR, TR, and WCI, tell us:

$$(1-\beta)(u_R+1)+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R+1)\Big)+\beta g$$

and

$$(1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(u_R-1)\Big)+\beta g \succ (1-\beta)(u_R-1)+\beta g$$

$$\Rightarrow (1-\beta)\frac{1}{4}(u_R+1)+\beta g \succeq (1-\beta)\frac{1}{4}R+\beta g \succ (1-\beta)\frac{1}{4}(u_R-1)+\beta g, \tag{3}$$

and

$$(1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(l_R+1)\Big)+\beta g \succ \Big(\frac{1}{2}R+\frac{1}{2}(l_R+1)\Big)+\beta g$$

and

$$(1-\beta)\Big(\frac{1}{2}R+\frac{1}{2}(l_R-1)\Big)+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(l_R-1)\Big)+\beta g$$

$$\Rightarrow (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_R+1)\Big)+\beta g \succ (1-\beta)\frac{1}{2}R+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(l_R-1)\Big)+\beta g. \quad (4)$$

(Again, this argument works unless $u_R = 100$, in which case COM, STR, and Axiom 6, tell us $(1-\beta)\frac{1}{4}u_R + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g \succ (1-\beta)\frac{1}{4}(u_R - 1) + \beta g$, in addition to $(1-\beta)(\frac{1}{4}R + \frac{1}{4}(l_R + 1)) + \beta g \succ (1-\beta)\frac{1}{2}R + \beta g \succeq (1-\beta)(\frac{1}{4}R + \frac{1}{4}(l_R - 1)) + \beta g$, and the rest of the argument becomes easier.)

Similarly, since the DM does not want to increase u_L or decrease l_L, COM, STR, CON, TR, and WCI, tell us:

$$(1-\beta)\Big(\frac{1}{2}R + \frac{1}{2}(u_L + 1)\Big) + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}(u_L + 1)\Big) + \beta g,$$

$$\text{and } (1-\beta)\frac{100}{2} + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}(l_L - 1)\Big) + \beta g, \quad (5)$$

$$\Rightarrow (1-\beta)\frac{1}{2}R + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{1}{4}(100 - u_L - 1)\Big) + \beta g, \quad (6)$$

$$\text{and } (1-\beta)\frac{1}{4}(100 - l_L + 1) + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g. \quad (7)$$

(This argument works unless $l_L = 0$, in which case COM, STR, and Axiom 6, tell us $(1-\beta)\frac{1}{2}R + \beta g \succeq (1-\beta)(\frac{1}{4}R + \frac{1}{4}(100 - u_L - 1)) + \beta g$, and $(1-\beta)\frac{1}{4}(100) + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g$, and the rest of the argument is easier.) Thus, equations (3) and (7), COM, STR, and TR, tell us:

$$(1-\beta)\frac{1}{4}(100 - l_L + 1) + \beta g \succ (1-\beta)\frac{1}{4}(u_R - 1) + \beta g \Rightarrow 101 \geq u_R + l_L.$$

Thus $l_L < u_L$ since $u_R + u_L \geq 102$. Next, notice that COM, STR, WCI, and the fact that the DM does not want to decrease u_L or increase l_L, tell us:

$$(1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}(u_L - 1)\Big) + \beta g \succ (1-\beta)\Big(\frac{1}{2}R + \frac{1}{2}(u_L - 1)\Big) + \beta g, \quad (8)$$

$$\text{and } (1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}(l_L + 1)\Big) + \beta g \succ (1-\beta)\frac{100}{2} + \beta g. \quad (9)$$

Thus, TR and WCI and equations (8), (6), (7), and (9), tell us:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(100-u_L+1)\Big)+\beta g \succ (1-\beta)\frac{1}{2}R+\beta g \succeq (1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}(100-u_L-1)\Big)+\beta g \quad (10)$$

$$\text{and } (1-\beta)\frac{1}{4}(100-l_L+1)+\beta g \succeq (1-\beta)\frac{1}{4}R+\beta g \succ (1-\beta)\frac{1}{4}(100-l_L-1)+\beta g. \quad (11)$$

So, using both equation (3) and equation (11) we get, using COM, STR, and TR, $u_R + l_L \in [99, 101]$. Next, using both equation (4) and equation (10) we get, using COM, STR, and TR:

$$(1-\beta)\frac{1}{4}(R + l_R + 1) + \beta g \succ (1-\beta)\frac{1}{4}(R + 100 - u_L - 1) + \beta g,$$

and

$$(1-\beta)\frac{1}{4}(R + 100 - u_L + 1) + \beta g \succ (1-\beta)\frac{1}{4}(R + l_R - 1) + \beta g,$$

$$\Rightarrow u_L + l_R \in [99, 101].$$

Case 5: Finally, if $u_R = u_L = u$ (which implies $u \geq 51$), then if both intervals are non-degenerate COM, STR, WCI, and the upper bounds, imply (since the DM does not lower both upper bounds):

$$(1-\beta)\Big(\frac{1}{2}\frac{100}{2} + \frac{1}{2}(u-1)\Big) + \beta g \succ (1-\beta)(u-1) + \beta g.$$

This results in a contradiction with COM and STR immediately. If, instead, both intervals are degenerate then similarly, COM,STR, WCI, tells us the DM could do strictly better by decreasing all four bounds by one since $u \geq 51$.

Further, if one interval is degenerate and one is non-degenerate, then assume without loss of generality (given $u_R = u_L = u$) that $l_L < u$. Then, COM, STR, WCI, and the fact that the DM does not benefit from increasing u_R tells us (using WCI):

$$(1-\beta)(u+1) + \beta g \succeq (1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(u+1)\Big) + \beta g$$

$$\Rightarrow (1-\beta)\frac{1}{4}(u+1) + \beta g \succeq (1-\beta)\frac{1}{4}R + \beta g,$$

COM, STR, WCI, and the fact that the DM does not change l_L, thus tell us (using COM, STR, TR, and WCI):

$$(1-\beta)\Big(\frac{1}{4}(l_L + 1) + \frac{1}{2}\frac{100}{2} + \frac{1}{4}R\Big) + \beta g \succ (1-\beta)\frac{100}{2} + \beta g$$

$$\Rightarrow u_R + l_L \geq 99,$$

$$\text{and } (1-\beta)\frac{100}{2} + \beta g \succeq (1-\beta)\Big(\frac{1}{4}(l_L - 1) + \frac{1}{2}\frac{100}{2} + \frac{1}{4}R\Big) + \beta g.$$

(This argument works unless $l_L = 0$, in which case $u_R + l_L \leq 101$ in addition to $u_R + l_L \geq 99$ and we are done.) COM, STR, WCI, and the fact that the DM does not benefit from lowering l_R, tell us (using COM, STR, TR, and WCI):

$$(1-\beta)\Big(\frac{1}{4}R + \frac{1}{4}(u-1) + \frac{1}{2}\frac{100}{2}\Big) + \beta g \succ (1-\beta)\Big(\frac{1}{2}(u-1) + \frac{1}{2}\frac{100}{2}\Big) + \beta g$$

$$\Rightarrow (1-\beta)\frac{1}{4}R + \beta g \succ (1-\beta)\frac{1}{4}(u-1) + \beta g \Rightarrow u_R + l_L \leq 101.$$

(It cannot be that $(1-\beta)(\frac{1}{4}R + \frac{1}{4}(u-1) + \frac{1}{2}\frac{100}{2}) + \beta g \sim (1-\beta)(\frac{1}{2}(u-1) + \frac{1}{2}\frac{100}{2}) + \beta g$ because then the DM could strictly benefit from either lowering l_R or from lowering all three of u_R, u_L, and l_R by one by COM, STR, TR, and WCI.) ■

Theorem 4.1 If the preferences of the DM satisfy WP, COM, WSTR, CON, TR, CI, and Axiom 6, and they answer all three questions in conjunction and assign equal and strictly positive weights to the probability equivalent questions, then $l_L \geq u_L - 1$, $l_R \geq u_R - 1$, $u_L + u_R \leq 101$, and $l_L + l_R \geq 99$.

Proof. Let $\alpha \in \{0, 0.01, 0.02, \ldots, 1\}$ be the chance that the DM selects to assign to betting on R as opposed to L in the randomization question. Let $\beta \in [0, 1)$ denote the DM's weight on the randomization question when they answer the probability equivalent questions. Lemma 4 tells us there is $x_L \in X$ and $x_R \in X$ such that $L \sim x_L$ and $R \sim x_R$. COM, STR (which is satisfied by Lemma 4), and CI, tell us to consider $\max(x_L, x_R) > 50$, because otherwise $l_L = u_L = l_R = u_R = 50$ and we are done. It is thus without loss to assume $\max(x_L, x_R) = x_R > 50$, and thus WP, COM, CON, and TR tell us $x_L \leq 50$, so $\alpha = 1$ if $\beta > 0$, $u_R \geq u_L$, and $l_R \geq l_L$.

Assume that the DM did report an interval of size two or more and we will reach a contradiction. Lemma 8 thus tells us $u_R \geq 51$. If $u_R = u_L$ then $l_R = u_R$ otherwise COM and STR tell us the DM could strictly benefit from lowering u_R and u_L by one, but then COM, STR, TR, and CI tell us the DM should lower u_L by one. So, $u_R > u_L$. If $l_R < u_R$ then the fact that the DM does not lower u_R, COM, STR, and WCI, tell us:

$$(1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}(u_R - 1)\Big) + \beta R \succ (1-\beta)(u_R - 1) + \beta R$$

$$\Rightarrow (1-\beta)\frac{1}{4}R + \beta R \succ (1-\beta)\frac{1}{4}(u_R - 1) + \beta R \Rightarrow x_R > u_R - 1,$$

but then Lemma 7, COM, STR, TR, and CI, tell us $l_R \geq u_L$ because if $l_R < u_L$ then $l_R + 1 \leq u_R - 1 < x_R$ and increasing l_R (since Lemma 7 tells us $l_L \leq l_R$) changes the chosen act for $y \in [l_R,\, l_R + 1)$ from:

$$(1-\beta)\Big(\frac{1}{2}\frac{100}{2} + \frac{1}{2}y\Big) + \beta R$$

to:

$$(1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}y\Big) + \beta R,$$

which then strictly benefits them and thus, also, $l_R \geq u_R - 1$ because if $l_R < u_R - 1$ then $l_R + 1 \leq u_R - 1 < x_R$ and increasing l_R changes the chosen act for $y \in (l_R,\, l_R + 1)$ from:

$$(1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}y\Big) + \beta R$$

to:

$$(1-\beta)\Big(\frac{1}{2}R + \frac{1}{2}y\Big) + \beta R,$$

and changes the chosen act for $y = l_R$, depending on if $l_R > u_L$ or $l_R = u_L$, from:

$$(1-\beta)\Big(\frac{1}{4}R + \frac{3}{4}l_R\Big) + \beta R$$

to:

$$(1-\beta)\Big(\frac{1}{2}R + \frac{1}{2}l_R\Big) + \beta R,$$

or from:

$$(1-\beta)\Big(\frac{1}{2}\frac{100}{2} + \frac{1}{2}l_R\Big) + \beta R$$

to:

$$(1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}l_R\Big) + \beta R,$$

and thus since one interval was assumed to be of size two or more we require $l_L \leq u_L - 2$, and thus to prevent the DM from strictly benefiting from either lowering u_L or increasing l_L COM, STR, and CI, tell us we require:

$$(1-\beta)\Big(\frac{1}{4}x_R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}(u_L - 1)\Big) + \beta x_R \sim (1-\beta)\Big(\frac{1}{4}R + \frac{1}{2}\frac{100}{2} + \frac{1}{4}(u_L - 1)\Big) + \beta R$$

$$\succ (1-\beta)\Big(\frac{1}{2}R + \frac{1}{2}(u_L - 1)\Big) + \beta R \sim (1-\beta)\Big(\frac{1}{2}x_R + \frac{1}{2}(u_L - 1)\Big) + \beta x_R,$$

and

$$(1-\beta)\Big(\frac{1}{4}x_R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(l_L+1)\Big)+\beta x_R \sim (1-\beta)\Big(\frac{1}{4}R+\frac{1}{2}\frac{100}{2}+\frac{1}{4}(l_L+1)\Big)+\beta R$$

$$\succ (1-\beta)\frac{100}{2}+\beta R \sim (1-\beta)\frac{100}{2}+\beta x_R,$$

which contradicts COM, STR, TR, and CI since then:

$$(1-\beta)\Big(\frac{100}{4}\Big) > (1-\beta)\Big(\frac{1}{4}x_R+\frac{1}{4}(u_L-1)\Big) \text{ and } (1-\beta)\Big(\frac{1}{4}x_R+\frac{1}{4}(l_L+1)\Big) > (1-\beta)\frac{100}{4}.$$

If, instead, $l_R = u_R$, then once again $l_L \leq u_L - 2$, and considering the conditions required for the DM to not strictly benefit from lowering u_L or increasing l_L above, COM, STR, TR, and CI, create the same contradiction.

Further, Theorem 3.1 then tells us that $u_R + u_L \leq 102$, and if $u_R + u_L = 102$ it cannot be that $u_R = u_L = 51$ as then $\min(l_L, l_R) \leq 50$ and COM, STR, TR, and CI, tell us the DM can do strictly better by either lowering u_L by one if $\max(l_L, l_R) = 51$ (remember $l_R \geq l_L$) or by decreasing both u_R and u_L by one if $\max(l_L, l_R) \leq 50$, so if $u_R + u_L = 102$ it must then be that $u_L \leq u_R - 2$, $u_L = l_L + 1$, $u_R = l_R + 1$, and since we thus have $u_L \leq l_R$, then $u_L < l_R$, $u_L \leq 50$, $u_R \geq 52$, and the fact that the DM does not lower u_R, COM, STR, TR, and WCI, tell us:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{3}{4}(l_R)\Big)+\beta R \succ (1-\beta)(l_R)+\beta R \Rightarrow x_R > l_R = 100 - l_L,$$

and COM, STR, TR, and CI, tell us the DM should lower u_L, so $u_L + u_R \leq 101$.

Theorem 3.1 also then tells us that $l_R + l_L \geq 98$, and if $l_R + l_L = 98$ it cannot be that $l_R = l_L = 49$ as then $\max(u_L, u_R) \geq 50$ and COM, STR, TR, and CI, tell us the DM can do strictly better by either increasing l_R by one if $\min(u_L, u_R) = 49$ (remember $u_R \geq u_L$) or by increasing both l_R and l_L by one if $\min(u_L, u_R) \geq 50$, so if $l_R + l_L = 98$ it must then be that $l_L \leq l_R - 2$, $u_L = l_L + 1$, $u_R = l_R + 1$, and since $u_L \leq l_R$, then $u_L < l_R$, $l_L \leq 48$, $l_R \geq 50$, and the fact that the DM does not increase l_L, COM, STR, TR, and WCI, tell us:

$$(1-\beta)\Big(\frac{1}{4}R+\frac{1}{4}\frac{100}{2}+\frac{1}{4}(u_L)\Big)+\beta R \succ (1-\beta)(\frac{100}{2})+\beta R \Rightarrow x_R > 100-u_L = u_R,$$

and COM, STR, TR, and CI, tell us the DM should increase l_R, so $l_L + l_R \geq 99$. ■

C.3 Additional Experimental Details

This subsection provides more details on the experiment conducted by Halevy et al. (2023). Any mention of "the" experiment or "our" experiment refers to the experiment conducted by Halevy et al. (2023).

C.3.1 Lottery Rounds

The three questions used in the "Lottery" rounds are described in Section 4.2 of Halevy et al. (2023). The interface in an example Lottery round can be found by clicking here.

To answer questions 1 and 2 the subjects use double-sliders similar to the ones in the Big-Shape rounds. The initial position of the double-sliders for both shapes are 0 and 100 for the lower and upper bound respectively. If question 1 or 2 above is used for payment then the double-slider response of the subject is compared to a random lottery r between 0 and 100. If r is below both sliders then the subject bets on the shape that is having its relative size elicited, which means they win the \$30 prize with a chance that is equal to the percent of the circle that is covered by the shape, if r is above both sliders then they bet on r, which means they win the prize with an $r\%$ chance, and if r is equal to or between the sliders then they bet on the shape or r with equal chances.

To answer question 3, subjects use a single slider to set the color composition of an urn which contains 100 balls, just like in the Big-Shape rounds, but with different consequences. The initial position of the slider for the urn is uniformly distributed over $\{0, 1, \ldots, 100\}$ and is recorded in our data. Suppose they set the slider so that the urn contains x blue balls, and consequently $100 - x$ red balls. If the question is used for payment there is a $x\%$ chance they bet on the left shape and win the prize with a chance that is equal to the percent of the left circle that is covered by the left shape, and there is a $(100 - x)\%$ chance they bet on the right shape and and win the prize with a chance that is equal to the percent of the circle that is covered by the right shape.

In all 12 of the Lottery rounds the subject gets to view the image of the shapes for the round, and the images that they see in the Lottery rounds are the images that they see in the Big-Shape rounds with the exception of the image that is suppressed for the subject in the Big-Shape round where they do not see an image.

We have complete process data for all 5 sliders in each round. We see when each change of each slider is made, and from what position to what position it is moved.

C.3.2 Main Order Treatments

As is mentioned in Section 4 of Halevy et al. (2023), we have four main order treatments, with subjects randomly distributed among them in a two-by-two factorial design. The two dimensions we vary are whether the subjects complete the Big-Shape or Lottery rounds first, and whether the three questions within each type of round are ordered such that the randomization question appears at the top or at the bottom of the screen. These treatments are meant to control for order effects among the different questions and among the different tasks, but any order effects are inconsequential to our main findings as is evident from Table 9, which describes behavior in the Big-Shape rounds. The training and quiz questions for each of these four main treatments are tailored to the specific treatment, i.e. the number and order of the three questions in the training and the order of the trainings and quizzes differ.

Table 9: Behavior Conditional On Treatment

	Treatment 1	Treatment 2	Treatment 3	Treatment 4
Subjects that violate Theorem 2.1	78% (51/65)	84% (48/57)	69% (34/49)	74% (35/47)
Rounds that violate Theorem 2.1	42%	36%	31%	33%
Rounds with randomization over L and R	64%	61%	64%	62%
Rounds with a non-degenerate interval	44%	42%	34%	41%

Treatment 1: Big-shape rounds first, randomization question at the bottom; Treatment 2: Big-shape rounds first, randomization question the top. Treatment 3: Lottery rounds first, randomization question at the top. Treatment 4: Lottery rounds first, randomization question at the top. The percent of subjects and rounds that are inconsistent with generalized Variational Preferences (violate Theorem 2.1), the percent of rounds with randomization over betting on the left and right shape, and the percent of rounds with a non-degenerate interval, are roughly the same in all the 4 main order treatments.

C.3.3 Images and Order

As mentioned above, for each subject we use the same set of 12 comparisons between shapes for both the Big-Shape and Lottery rounds. Each comparison has a proper name which uniquely identifies it regardless of the order in which it is seen by the subjects: Image 1, Image 2, Image 2B, Image 3, Image 4, Image 4B, Image 5, Image 6, Image 7, Image 8, Image 9, Image 10, Image 10B, Image 11, and Image 12, most of which can be found in Figure 9. Comparisons differ from each other in several ways. Some include comparisons between squares, while others include comparisons between rectangles, while Image 7, for example, includes a comparison between a cross and a square. Moreover, for the majority of comparisons, we have removed a small slice from both shapes in order to make it more challenging for subjects to measure the objects on their computer using a ruler.[40]

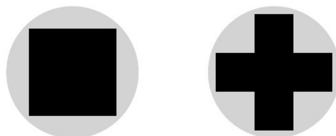
Exceptionally, the first comparison faced by subjects in the Big-Shape rounds does not include any images, yet subjects are still asked to respond to the three questions. This provides a sanity check, to evaluate if the subject reduces compound lotteries (as discussed in Online Appendix B in (Halevy et al., 2023) and also allows us to detect an inherent desire to randomize which is independent of the characteristics of the choice objects, i.e. subjects may just like to randomize for the sake of it as implied by the model of Fudenberg et al. (2015). In that model a DM can strictly benefit from randomizing over options even if they are identical, could randomize over options that do not provide the same value to the DM as randomization for randomization's sake is valuable, and could thus choose a non-degenerate belief interval even if they posses a degenerate 'point' belief about the state of the world. In our experiment, however, 70% of subjects listed degenerate intervals of 50 as their probability equivalents for both the left and right shape when faced with the Big-Shape round with no image, and thus they did not demonstrate a desire to randomize for the sake of it in general.

The twelve types of images are grouped in ordered blocks of three, called A (Image 1, Image 2 or Image 8, and Image 3), B (Image 4 or Image 4B, Image 5, and Image 6), C (Image 7, Image 8 or Image 2B, and Image 9), and D (Image 10 or Image 10B, Image 11, and Image 12) respectively. Subjects see one of four possible orderings of the blocks: ABCD, BCDA, CDAB, or DABC. These different block orderings allow us to partially control for order effects among the comparisons, but we do not see any substantial evidence of order effects. We chose to change the order in blocks so as to avoid two adjacent comparisons that may appear too similar, leaving the subject with the impression that they are repeating the same comparisons over and over. While it is true that some of the comparisons are, in fact, very similar, there are no two that

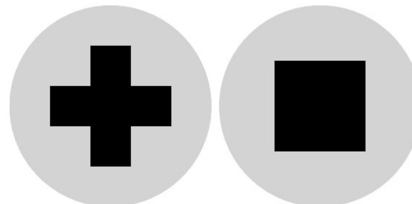
[40]There exists software that allows for users to measure the areas fairly precisely. Even so, any subject who uses this software would be straightforward to identify, hence we do not consider this to be a significant concern.

Figure 9: Images Used in the Experiment

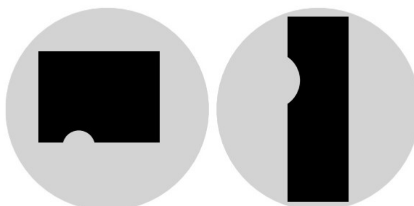
(a) Image 1



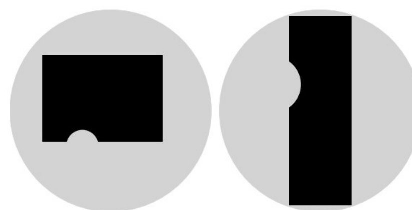
(b) Image 7



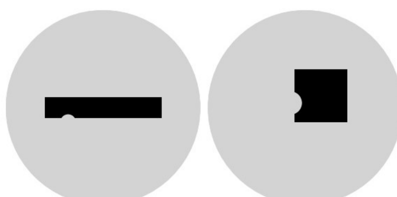
(c) Image 3



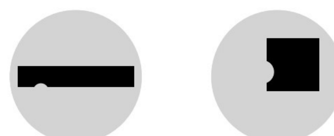
(d) Image 9



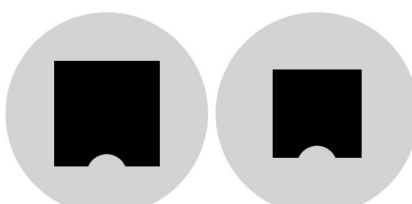
(e) Image 4



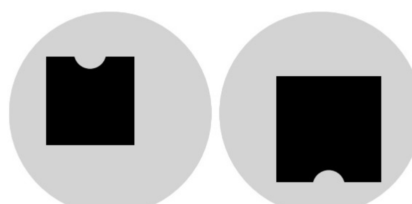
(f) Image 10



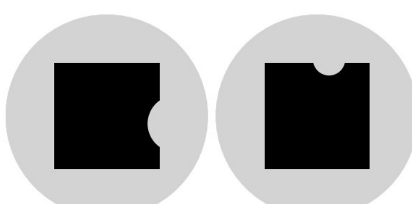
(g) Image 5



(h) Image 11

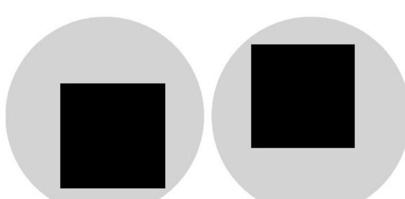


(i) Image 6

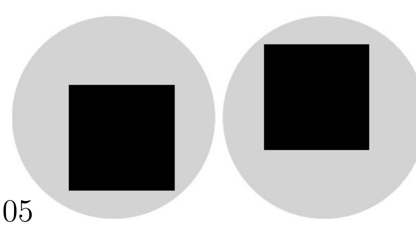


(j) Image 12

(k) Image 2



(l) Image 2B



are identical in either type of round, even if they appear so.

In order to detect any potential bias towards the image on the right or left side, we also exchange both Image 4 and Image 10 for both Image 4B and Image 10B for about half of the subjects, which are 180 degree rotations of Image 4 and Image 10. We do not see evidence of a substantial bias towards one side. Finally, some subjects see Image 2 that has the same shape areas as Image 8 (but which side has the larger shape is switched) while others see Image 2B that is slightly different than Image 2. Subjects that see Image 2 have it in block A and Image 8 in block C, whereas subjects that see Image 2B have it in block C and Image 8 in block A. This allows us to attempt to hold difficulty fixed while varying the dimensionality of the comparison, and test for substantial order effects (which we do not see evidence of).

There are thus four different image set compositions subjects see, and each composition has four potential block orderings (ABCD, BCDA, CDAB, or DABC), for a total of 16 different sequences of images. Along with the four main treatments, this means there are 64 possible subject experiences.

C.4 Experimental Interface: Instructions and Quizzes

The following pages are a sample of the instructions and quizzes that may be seen by subjects in the experiment conducted Halevy et al. (2023). There are 4 possible combinations, as detailed in Section C.3.2. What follows is a mixture that cannot be seen by any single subject but allows the reader to get a sense of all possible subject experiences. Here, we start with the Big-Shape rounds with the randomization question at the top, followed by the Lottery rounds with the randomization question at the bottom.

Note: while the sliders themselves are not visible in these screenshots (due to technical difficulties), they were visible to the subjects, as they are in Figures 6 and 7 in the work of Halevy et al. (2023).

Consent Form

Who is conducting the study?
The principal investigators are Yoram Halevy, David Walker-Jones, and Lanny Zrill.

Who is funding this study?
This study is run with external funds from SSHRC administered by the University of Toronto.

Why are we doing this study?
Making choices under uncertainty when information about the consequences of one's choices is available plays a central role in economic models. The purpose of this study is to observe how decisions are made under different information and feedback conditions.

How is the study done?
In this study you will be asked to make many decisions about how confident you are in your perception of which of two shapes is larger. We expect the experiment to take up to 60 minutes to complete, but you can take as much time as you want to finish it. Please note that, as in all experiments in economics, there is no deception in the study: we are not trying to trick you, and all payments are real.

What will we do with the study results?
We expect that the results of the study will be published in academic journals.

Could participating in the study be bad for you?
We do not think there is anything in this study that could harm you or be bad for you.

What are the benefits of participating?
You will be paid for your participation in the study (see below). In addition, the decisions you will be asked to make could be interesting, and further, you may learn something about different learning heuristics.

How will your privacy be maintained?
Your performance and decisions will be kept strictly confidential. You will never be identified by name or any other identifying feature with relation to this study. All electronic data from this experiment will be analyzed anonymously and will be kept on a password protected secure server.

Will you be paid for taking part in the study?
As long as you have online banking with a Canadian bank so that you can receive Interac e-transfers, you will be paid between $5 and $10 for completing the training portion of the experiment, and in addition the experiment gives you the chance to win a monetary prize once you are done with the experiment. You will receive all payments as Interac e-transfers after the experiment.

Who can you contact if you have questions about this study?
If you have any questions or concerns, or desire additional information about this study, please contact David Walker-Jones at david.walker.jones@mail.utoronto.ca, or Yoram Halevy at yoram.halevy@utoronto.ca.

Who can you contact if you have complaints or concerns about the study?
If you have any concerns or complaints about your rights as a research participant and/or your experiences while participating in this study, contact the Office of Research Ethics at ethics.reviews@utoronto.ca or 416-946-3273.

Consent: Information / feedback Taking part in this study is entirely up to you. You have the right to refuse to participate in this study. If you decide to take part, you may choose to pull out of the study at any time without giving a reason and without any negative impact on your class standing. When you type in the field below 'I consent', it serves as an electronic signature and indicates that you consent to participate in this experiment. If you wish, you can print this page for your own records.

Please enter your age in years:

Please enter the day of the month:

Please select the month:

Please select the year:

Please enter your student number:

Please enter the email with which you would like to receive the Interac e-transfer of any money you earn:

Please enter your full name:

If you wish to consent to participating in the experiment, please type 'I consent':

Next

Welcome to this TEEL online experiment!

Thank you for participating in this study of confidence in choice under uncertainty. **Please read the following instructions carefully**. Understanding what is going on could help you earn more money (**you can earn up to $40 in this experiment**).

The experiment is divided into two sections which have 12 rounds of questions each. Each section begins with some training and a quiz to confirm your understanding. There will be 5 questions in each quiz. You must answer each question correctly before you can move on to the next question. When you complete each quiz you will earn a $2.50 training payment. In addition, for each quiz question you answer correctly on the first attempt you will earn another $0.5 training bonus payment once you finish the quiz. This means **you can earn up to $10 just by doing well during the training.**

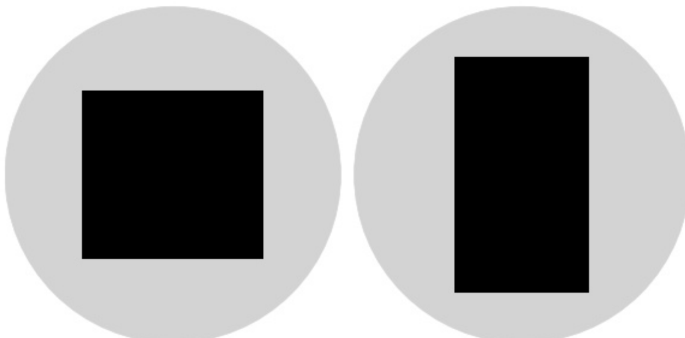
When you click the 'Next' button and move on from one page to the next you will not be able to go back, so **make sure you read everything carefully before moving on**. Please do not try to use the back button or the refresh button during the experiment; this could cause the experiment to crash.

You have until 6 PM Toronto time (EST) to complete the experiment. If you have questions or require any kind of assistance please use the Zoom link that was included in the invitation you received to participate and we will be happy to help.

Next

Training Part 1 (page 1 of 3):

The experiment consists of **24 rounds.** Each round has an **image of two circles with shapes inside** of them. **Below is an example**:



In each round one of the two shapes inside of the circles is larger (has the bigger area), either the shape in the left circle (the "left shape" for short) or the shape in the right circle (the "right shape" for short). The left circle and right circle are the same size as each other in each round. Even if the images in two rounds look similar it does not imply that they are the same image.

Your goal is to try to determine the relative sizes of the two shapes and how confident you are in your judgement. Choosing the larger shape and deciding how confident you are that you chose correctly will increase your chance of winning the **$30 prize**, which is in addition to whatever you earn from the training. How the payment of the $30 prize is determined will be explained over the course of the training.

In each round, **the chance of the shape in the left circle being larger than the shape in the right circle is exactly 50%,** which is the same as the chance of the shape in the right circle being larger than the shape in the left circle.

In each round, except for Round 1, **we will show you the image of the shapes for the round**.

In each round, **you will answer 3 questions** about the relative sizes of the shapes. You must spend at least **45 seconds** answering these questions before you can move on to the next round. In Round 1 the image of the shapes is hidden from you, so you must answer the 3 questions without seeing the shapes.

The 3 questions that you answer in each of the first 12 rounds and the 3 questions that you answer in each of the last 12 rounds differ from each other slightly, however. You will be trained on the questions in the the last 12 rounds after you finish the first 12 rounds.

In the first 12 rounds you care about which of the two shapes in the circles is larger, and the 3 questions you answer are:
Question 1: Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options? (Betting on the larger shape increases your chance of winning the $30 prize, you will learn more about this soon)
Question 2: What do you think is the chance that the shape in the right circle is larger?
Question 3: What do you think is the chance that the shape in the left circle is larger?

Your chance of winning the $30 prize is determined by your answer to one of the 3 questions in one of the 24 rounds. This question, which we say is **used for payment**, has already been randomly selected by the computer, so your decisions do not impact which question is used for payment. You do not know which question is used for payment, however, so **you may answer each question as if it is the question being used for payment.**

Behind the scenes there is a **random number** between 0 and 100. The random number has already been randomly selected by the computer, so your decisions do not impact it.
When you answer questions 2 and 3 in each of the first 12 rounds you essentially tell us, for each potential value of the random number, if you would rather win the $30 prize if the shape in the relevant circle is larger or win the $30 prize with a percent chance that is equal to the random number. You will see the specifics of how all of this works in the coming pages.

If you want to stop the experiment before completing the 24 rounds then you have the **option to exit** below the "Next" button at the bottom of each page with the 3 questions for a round. If you exit before the question that is used for payment, however, then you will not be able win the $30 prize, and you will only get what you earned from the training.

Next

Training Part 1 (page 2 of 3):

Question 1 asks you if you would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options.

You can **randomize** by assigning each of **100 different balls** to betting on the left or right shape (you will see how this works below). **The computer will draw 1 of the 100 balls at random,** each with the same chance. If the drawn ball is red it places a bet for you on the right shape, and if the drawn ball is blue it places a bet for you on the left shape.

Question 1 is the same in all 24 rounds, but the result of betting on a shape is different in the first 12 rounds than in the last 12 rounds. If a Question 1 from the first 12 rounds is used for payment then you win the $30 prize if the drawn ball places a bet on the larger shape.

Remember: In each of the first 12 rounds, except for Round 1, you will be able to see the image of the shapes when you answer Question 1.

The first 12 rounds:

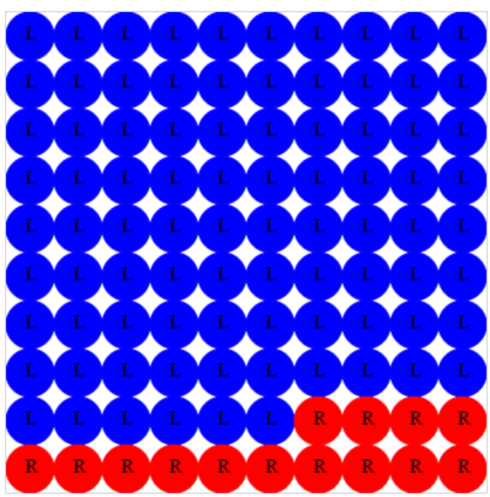
In each of the first 12 rounds you will answer Question 1 below. If such a question is used for payment, then you win the $30 prize if the ball that is drawn places a bet on the larger of the two shapes in the circles.

You answer the question using the slider below the diagram of the balls, and your answer appears just above the slider. As you move the slider the text above and below the slider will be updated. The text below the slider explains how your answer to the question determines your chance of winning the $30 prize if the question is used for payment.

The initial position of the slider for the question is randomly determined by the computer and has nothing to do with the image or how you have answered previous questions. The initial position of the slider is in no way meant to be a suggestion about how you should answer the question.

You should try moving the slider and see how the text above and below the slider changes.

Question 1: Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options?



I would like to assign 14 balls to betting on the shape in the right circle

and I would like to assign 86 balls to betting on the shape in the left circle

If this question is used for payment:

You have a 14% chance of betting on the shape in the right circle, in which case you win the $30 prize if the shape in the right circle is larger than the shape in the left circle and you do not win the prize if is not.

You have a 86% chance of betting on the shape in the left circle, in which case you win the $30 prize if the shape in the left circle is larger than the shape in the right circle and you do not win the prize if is not.

Before you move to the next page:
Answer Question 1 so that 27 balls are assigned to betting on the shape in the left circle.

Next

Training Part 1 (page 3 of 3):

In each of the first 12 rounds you will answer the following two questions:
Question 2: What do you think is the chance that the shape in the right circle is larger than the shape in the left circle?
Question 3: What do you think is the chance that the shape in the left circle is larger than the shape in the right circle?

You answer the questions using two double-sliders and your answer appears just above each double-slider. As you move each slider the text above and below the double-slider will be updated. When you answer a question the sliders can coincide (touch) if you want them to, or there can be a gap between the sliders if you want to report a range of confidences.

Remember: In each of the first 12 rounds, except for Round 1, you will be able to see the image of the shapes when you answer questions 2 and 3.

The text below each question explains how your answer to the question determines your chance of winning the $30 prize if the question is used for payment, and the "fair digital coin" has a 50% chance of coming up both heads and tails:

You should try moving the sliders and see how the text above and below each double-slider changes.

Question 2: What do you think is the chance that the shape in the **right** circle is larger?

The chance of the shape being larger is between 0% and 100%

☐ ☐

0 25 50 75 100

If the random number is below 0 and this question is used for payment:
You bet that the shape in the right circle is larger. This means that you win the $30 prize if the shape in the right circle is larger than the shape in the left circle and you do not win the prize if the shape in the right circle is smaller than the shape in the left circle.

If the random number is above 100 and this question is used for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 0 and 100 and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or bet that the shape in the right circle is larger.
If the coin comes up heads you bet that the shape in the right circle is larger: you win the $30 prize if the right shape is larger.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Question 3: What do you think is the chance that the shape in the **left** circle is larger?

The chance of the shape being larger is between 60% and 80%

☐ ☐

0 25 50 75 100

If the random number is below 60 and this question is used for payment:
You bet that the shape in the left circle is larger. This means that you win the $30 prize if the shape in the left circle is larger than the shape in the right circle and you do not win the prize if the shape in the left circle is smaller than the shape in the right circle.

If the random number is above 80 and this question is used for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 60 and 80 and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or bet that the shape in the left circle is larger.

If the coin comes up heads you bet that the shape in the left circle is larger: you win the $30 prize if the left shape is larger.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Before you move to the next page:
Answer Question 3 above so that if it were used for payment you would bet that the left shape is larger if the random number is smaller than 60, bet on the random number if the random number is larger than 80, and for all other values of the random number a fair digital coin is flipped for you that determines if you bet on the random number or bet that the shape in the left circle is larger.

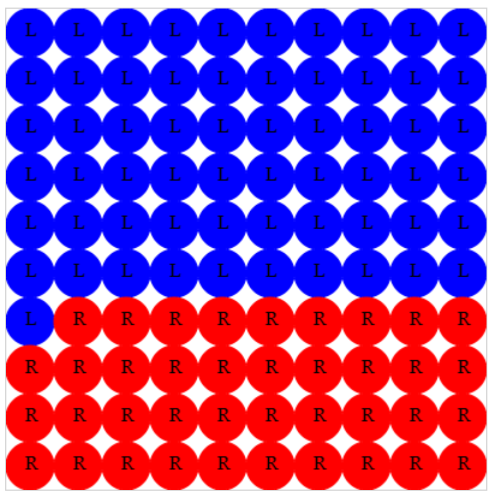
Next

Quiz time! Quiz 1, Question 1

It is now time to do the quiz. **Remember:** to move on to the next question you must first answer the current question correctly. By completing the quiz you will earn a $2.5 training payment, and in addition to the training payment, you can earn another $0.5 for each question you answer correctly on your first attempt.

If you are in one of the first 12 rounds and **you know for sure that the shape in the right circle is larger**, how should you answer Question 1, Question 2, and Question 3 below if one of them were used for payment and you want to maximize your chance of winning the $30 prize?

Question 1: Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options?



I would like to assign 39 balls to betting on the shape in the right circle

and I would like to assign 61 balls to betting on the shape in the left circle

If this question is used for payment:

You have a 39% chance of betting on the shape in the right circle, in which case you win the $30 prize if the shape in the right circle is larger than the shape in the left circle and you do not win the prize if is not.

You have a 61% chance of betting on the shape in the left circle, in which case you win the $30 prize if the shape in the left circle is larger than the shape in the right circle and you do not win the prize if is not.

Question 2: What do you think is the chance that the shape in the **right** circle is larger?

The chance of the shape being larger is between 0% and 100%

0 25 50 75 100

If the random number is below 0 and this question is used for payment:
You bet that the shape in the right circle is larger. This means that you win the $30 prize if the shape in the right circle is larger than the shape in the left circle and you do not win the prize if the shape in the right circle is smaller than the shape in the left circle.

If the random number is above 100 and this question is used for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 0 and 100 and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or bet the shape in the right circle is larger.
If the coin comes up heads you bet the shape in the right circle is larger: you win the $30 prize if the right shape is larger.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Question 3: What do you think is the chance that the shape in the **left** circle is larger?

The chance of the shape being larger is between 0% and 100%

☐ ☐

0 25 50 75 100

If the random number is below 0 and this question is used for payment:
You bet that the shape in the left circle is larger. This means that you win the $30 prize if the shape in the left circle is larger than the shape in the right circle and you do not win the prize if the shape in the left circle is smaller than the shape in the right circle.

If the random number is above 100 and this question is used for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 0 and 100 and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or bet the shape in the left circle is larger.
If the coin comes up heads you bet the shape in the left circle is larger: you win the $30 prize if the left shape is larger.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Next

Quiz time! Quiz 1, Question 2

If you are in one of the first 12 rounds and **you cannot see the image of the shapes for the round** how should you answer Question 2 and Question 3 below if either of them were used for payment and you want to maximize your chance of winning the $30 prize?

Question 2: What do you think is the chance that the shape in the **right** circle is larger?

The chance of the shape being larger is between 0% and 100%

☐ ☐

0 25 50 75 100

If the random number is below 0 and this question is selected for payment:
You bet that the shape in the right circle is larger. This means that you win the $30 prize if the shape in the right circle is larger than the shape in the left circle and you do not win the prize if the shape in the right circle is smaller than the shape in the left circle.

If the random number is above 100 and this question is selected for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 0 and 100 and this question is selected for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or bet the shape in the right circle is larger.
If the coin comes up heads you bet the shape in the right circle is larger: you win the $30 prize if the right shape is larger.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Question 3: What do you think is the chance that the shape in the **left** circle is larger?

The chance of the shape being larger is between 0% and 100%

☐ ☐

0 25 50 75 100

If the random number is below 0 and this question is selected for payment:
You bet that the shape in the left circle is larger. This means that you win the $30 prize if the shape in the left circle is larger than the shape in the right circle and you do not win the prize if the shape in the left circle is smaller than the shape in the right circle.

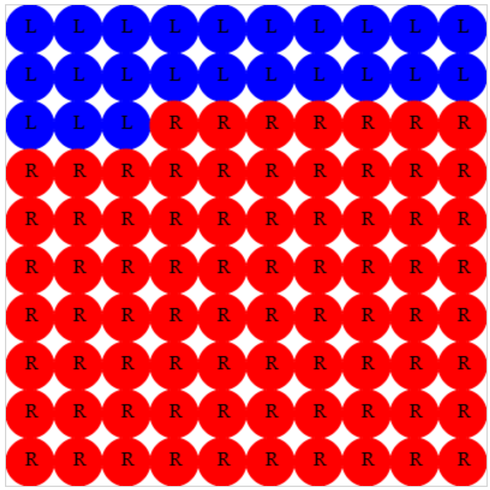
If the random number is above 100 and this question is selected for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 0 and 100 and this question is selected for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or bet the shape in the left circle is larger.
If the coin comes up heads you bet the shape in the left circle is larger: you win the $30 prize if the left shape is larger.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Next

Quiz time! Quiz 1, Question 3

Question 1: Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options?



I would like to assign 77 balls to betting on the shape in the right circle

and I would like to assign 23 balls to betting on the shape in the left circle

☐

If this question is used for payment:

You have a 77% chance of betting on the shape in the right circle, in which case you win the $30 prize if the shape in the right circle is larger than the shape in the left circle and you do not win the prize if is not.

You have a 23% chance of betting on the shape in the left circle, in which case you win the $30 prize if the shape in the left circle is larger than the shape in the right circle and you do not win the prize if is not.

If a Question 1 from the first 12 rounds like the one above is used for payment, which of the following is true about your chance of winning the $30 prize?

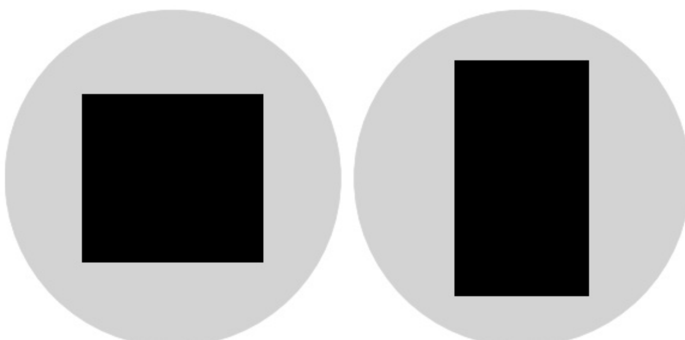
Next

Quiz time! Quiz 1, Question 4

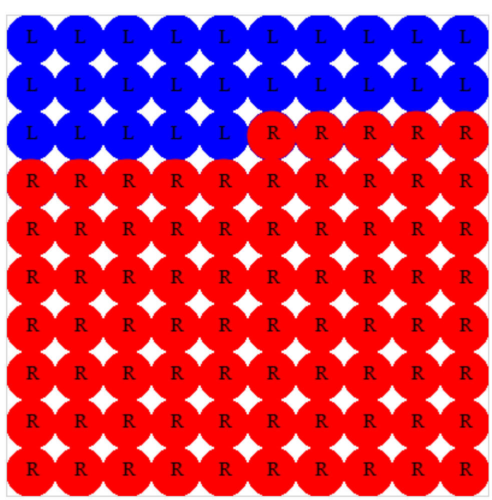
Please look at the image of shapes below and answer the questions about your confidence in which is larger:

Remember: you must spend at least 45 seconds on this page. The "Next" button will not appear until 45 seconds pass.

This round is for practice only. However, you will be rewarded for correct responses to the quiz questions on the following pages.



Question 1: Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options?



I would like to assign 75 balls to betting on the shape in the right circle

and I would like to assign 25 balls to betting on the shape in the left circle

If this question is used for payment:

You have a 75% chance of betting on the shape in the right circle, in which case you win the $30 prize if the shape in the right circle is larger than the shape in the left circle and you do not win the prize if is not.

You have a 25% chance of betting on the shape in the left circle, in which case you win the $30 prize if the shape in the left circle is larger than the shape in the right circle and you do not win the prize if is not.

Question 2: What do you think is the chance that the shape in the **right** circle is larger?

The chance of the shape being larger is between 45% and 60%

0 25 50 75 100

If the random number is below 45 and this question is used for payment:
You bet that the shape in the right circle is larger. This means that you win the $30 prize if the shape in the right circle is larger than the shape in the left circle and you do not win the prize if the shape in the right circle is smaller than the shape in the left circle.

If the random number is above 60 and this question is used for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 45 and 60 and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or bet the shape in the right circle is larger.
If the coin comes up heads you bet the shape in the right circle is larger: you win the $30 prize if the right shape is larger.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Question 3: What do you think is the chance that the shape in the **left** circle is larger?

The chance of the shape being larger is between 40% and 55%

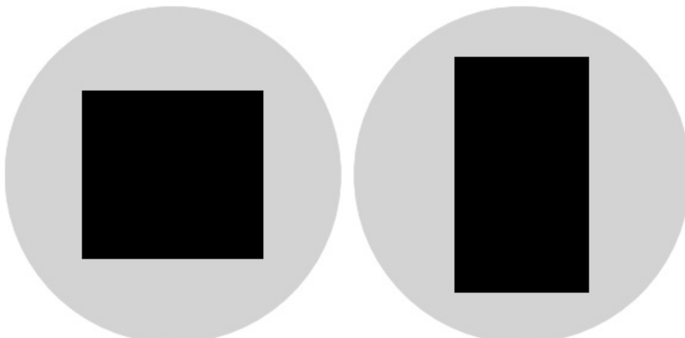
0 25 50 75 100

If the random number is below 40 and this question is used for payment:
you bet that the shape in the left circle is larger. This means that you win the $30 prize if the shape in the left circle is larger than the shape in the right circle and you do not win the prize if the shape in the left circle is smaller than the shape in the right circle.

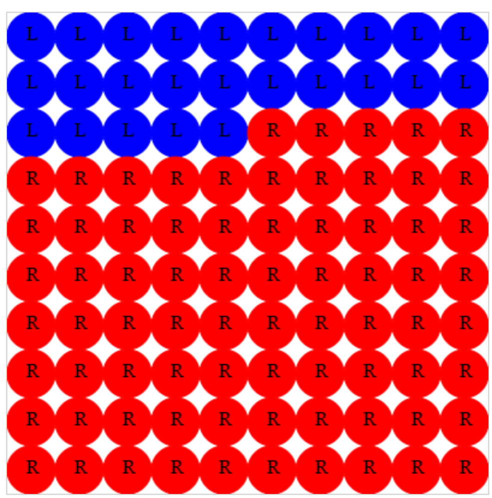
If the random number is above 55 and this question is used for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 40 and 55 and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or bet the shape in the left circle is larger.
If the coin comes up heads you bet the shape in the left circle is larger: you win the $30 prize if the left shape is larger.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Quiz time! Quiz 1, Question 4



Previously, you selected the following in response to the question: "Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options?"



I would like to assign 75 balls to betting on the shape in the right circle

and I would like to assign 25 balls to betting on the shape in the left circle

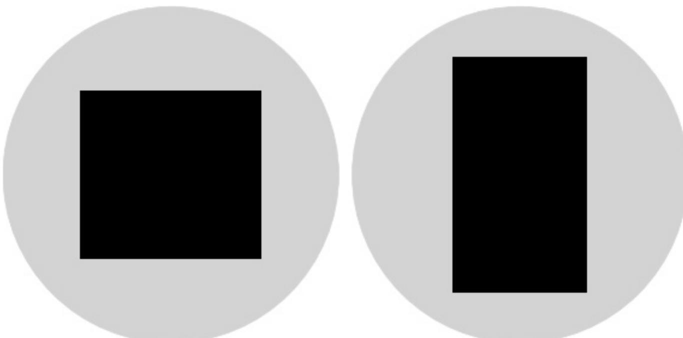
If this question is used for payment:

Suppose the shape in the right circle is bigger. In this case, you will have a 75% chance of winning the $30 prize as this is the chance that a red ball is randomly drawn.

On the other hand, suppose the shape in the left circle is bigger. In this case, what is the chance of winning the $30 prize?

Next

Quiz time! Quiz 1, Question 5



Previously, you selected the following in response to the question: "What do you think is the chance that the shape in the **right** circle is larger?"

The chance of the shape being larger is between 0% and 100%

☐ 0 25 50 75 100 ☐

If the random number is equal to or between 0 and 100, for example 39, and this question is used for payment, which of the following statements is true?

Next

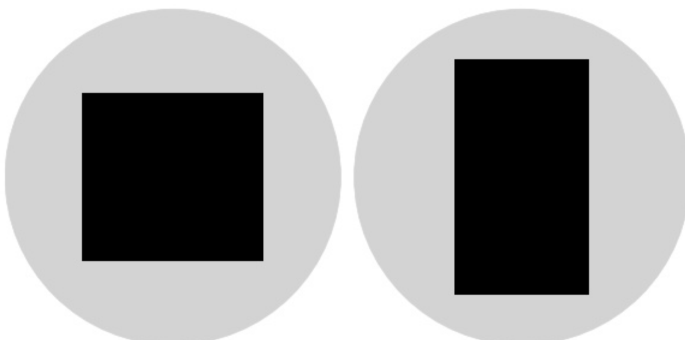
Congratulations: you have completed the quiz!

Up next: 12 rounds of perceptual decision problems.

Next

Training Part 1 (page 1 of 3):

The experiment consists of **24 rounds.** Each round has an **image of two circles with shapes inside** of them. **Below is an example**:



In each round one of the two shapes inside of the circles is larger (has the bigger area), either the shape in the left circle (the "left shape" for short) or the shape in the right circle (the "right shape" for short). The left circle and right circle are the same size as each other in each round. Even if the images in two rounds look similar it does not imply that they are the same image.

Your goal is to try to determine the relative sizes of the two shapes and how confident you are in your judgement. Choosing the larger shape and deciding how confident you are that you chose correctly will increase your chance of winning the **$30 prize**, which is in addition to whatever you earn from the training. How the payment of the $30 prize is determined will be explained over the course of the training.

In each round, **the chance of the shape in the left circle being larger than the shape in the right circle is exactly 50%,** which is the same as the chance of the shape in the right circle being larger than the shape in the left circle.

In each of the first 12 rounds **we will show you the image of the shapes for the round**.

In each round, **you will answer 3 questions** about the relative sizes of the shapes. You must spend at least **45 seconds** answering these questions before you can move on to the next round.

The 3 questions that you answer in each of the first 12 rounds and the 3 questions that you answer in each of the last 12 rounds differ from each other slightly, however. You will be trained on the questions in the the last 12 rounds after you finish the first 12 rounds.

In the first 12 rounds you care about how much of the circles are covered by the shapes, and the 3 questions you answer are:
Question 1: What percent of the right circle do you think is covered by the right shape?
Question 2: What percent of the left circle do you think is covered by the left shape?
Question 3: Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options? (Betting on the larger shape increases your chance of winning the $30 prize, you will learn more about this soon)

Your chance of winning the $30 prize is determined by your answer to one of the 3 questions in one of the 24 rounds. This question, which we say is **used for payment**, has already been randomly selected by the computer, so your decisions do not impact which question is used for payment. You do not know which question is used for payment, however, so **you may answer each question as if it is the question being used for payment.**

Behind the scenes there is a **random number** between 0 and 100. The random number has already been randomly selected by the computer, so your decisions do not impact it.
When you answer questions 1 and 2 in each of the first 12 rounds you essentially tell us, for each potential value of the random number, if you would rather win the $30 prize with a chance that is equal to the percent of the relevant circle that is covered by its shape or win the $30 prize with a percent chance that is equal to the random number. You will see the specifics of how all of this works in the coming pages.

If you want to stop the experiment before completing the 24 rounds then you have the **option to exit** below the "Next" button at the bottom of each page with the 3 questions for a round. If you exit before the question that is used for payment, however, then you will not be able win the $30 prize, and you will only get what you earned from the training.

Next

Training Part 1 (page 2 of 3):

In each of the first 12 rounds you answer the following two questions:
Question 1: What percent of the right circle do you think is covered by the right shape?
Question 2: What percent of the left circle do you think is covered by the left shape?

You answer the questions using two double-sliders, and your answer appears just above each double-slider. As you move each slider the text above and below the double-slider will be updated. When you answer a question the sliders can coincide (touch) if you want them to, or there can be a gap between the sliders if you want to report a range of percentages.

Remember: In each of the first 12 rounds you will be able to see the image of the shapes when you answer questions 1 and 2.

The text below each question explains how your answer to the question determines your chance of winning the $30 prize if the question is used for payment, and the "fair digital coin" has a 50% chance of coming up both heads and tails:

You should try moving the sliders and see how the text above and below each double-slider changes.

Question 1: What percent of the **right** circle do you think is covered by the shape in the **right** circle?

The percent of the circle covered is between 0% and 100%

0 25 50 75 100

If the random number is below 0 and this question is used for payment:
You bet on the percent of the right circle covered by its shape. This means that you win the $30 prize with a chance that is equal to the percent of the right circle that is covered by its shape.

If the random number is above 100 and this question is used for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 0 and 100 and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or on the percent of the right circle covered by its shape.
If the coin comes up heads you bet on the percent of the right circle covered by its shape: You win the $30 prize with a chance that is equal to the percent of the right circle that is covered by its shape.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Question 2: What percent of the **left** circle do you think is covered by the shape in the **left** circle?

The percent of the circle covered is between 0% and 100%

0 25 50 75 100

If the random number is below 0 and this question is used for payment:
You bet on the percent of the left circle covered by its shape. This means that you win the $30 prize with a chance that is equal to the percent of the left circle that is covered by its shape.

If the random number is above 100 and this question is used for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 0 and 100 and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or on the percent of the left circle covered by its

shape.
If the coin comes up heads you bet on the percent of the left circle covered by its shape: You win the $30 prize with a chance that is equal to the percent of the left circle that is covered by its shape.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Before you move to the next page:
Answer Question 1 above so that if it were used for payment you would bet on the percent of the right circle covered by its shape if the random number is smaller than 30, bet on the random number if the random number is larger than 35, and for all other values of the random number a fair digital coin is flipped for you that determines if you bet on the random number or bet on the percent of the right circle covered by its shape.

Next

Training Part 1 (page 3 of 3):

Question 3 asks you if you would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options.

You can **randomize** by assigning each of **100 different balls** to betting on the left or right shape (you will see how this works below). **The computer will draw 1 of the 100 balls at random,** each with the same chance. If the drawn ball is red it places a bet for you on the right shape, and if the drawn ball is blue it places a bet for you on the left shape.

Question 3 is the same in all 24 rounds, but the result of betting on a shape is different in the first 12 rounds than in the last 12 rounds. If a Question 3 from the first 12 rounds is used for payment, then you win the $30 prize with a chance that is equal to the percent of the circle that is covered by the shape that the drawn ball places a bet on.

Remember: In each of the first 12 rounds you will be able to see the image of the shapes when you answer Question 3.

The first 12 rounds:

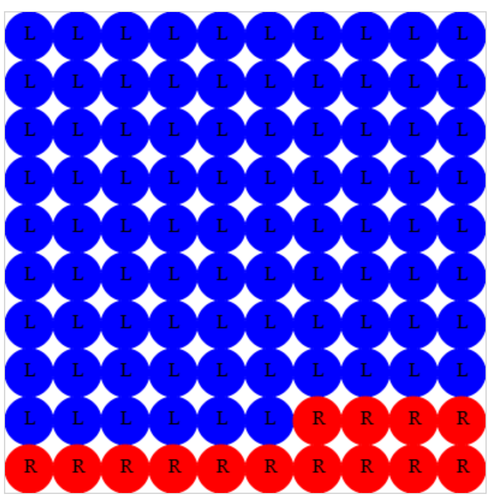
In each of the first 12 rounds you will answer Question 3 below. If such a question is used for payment, then you win the $30 prize with a chance that is equal to the percent of the circle that is covered by the shape that the drawn ball places a bet on.

You answer the question using the slider below the diagram of the balls, and your answer appears just above the slider. As you move the slider the text above and below the slider will be updated. The text below the slider explains how your answer to the question determines your chance of winning the $30 prize if the question is used for payment.

The initial position of the slider for the question is randomly determined by the computer and has nothing to do with the image or how you have answered previous questions. The initial position of the slider is in no way meant to be a suggestion about how you should answer the question.

You should try moving the slider and see how the text above and below the slider changes.

Question 3: Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options?



I would like to assign 14 balls to betting on the shape in the right circle

and I would like to assign 86 balls to betting on the shape in the left circle

If this question is used for payment:

You have a 14% chance of betting on the shape in the right circle, in which case you win the $30 prize with a chance that is equal to the percent of the right circle that is covered by the shape in the right circle.

You have a 86% chance of betting on the shape in the left circle, in which case you win the $30 prize with a chance that is equal to the percent of the left circle that is covered by the shape in the left circle.

Before you move to the next page:
Answer Question 3 so that 27 balls are assigned to betting on the shape in the left circle.

Next

Quiz time! Quiz 1, Question 1

It is now time to do the quiz. **Remember:** to move on to the next question you must first answer the current question correctly. By completing the quiz you will earn a $2.5 training payment, and in addition to the training payment, you can earn another $0.5 for each question you answer correctly on your first attempt.

Question 1: What percent of the **right** circle do you think is covered by the shape in the **right** circle?

The percent of the circle covered is between 55% and 65%

0 25 50 75 100

If the random number is below 55 and this question is used for payment:
You bet on the percent of the right circle covered by its shape. This means that you win the $30 prize with a chance that is equal to the percent of the right circle that is covered by its shape.

If the random number is above 65 and this question is used for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 55 and 65 and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or on the percent of the right circle covered by its shape.
If the coin comes up heads you bet on the percent of the right circle covered by its shape: You win the $30 prize with a chance that is equal to the percent of the right circle that is covered by its shape.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

If a Question 1 from the first 12 rounds is used for payment, and you answer it as is done above, what is true about your probability of winning the $30 prize?

Next

Quiz time! Quiz 1, Question 2

If you are in one of the first 12 rounds and you know for sure that the shape in the right circle takes up 35% of its circle and the shape in the left circle takes up 60% of its circle, how should you answer Question 1, Question 2, and Question 3 below if one of them were used for payment and you want to maximize your chance of winning the $30 prize?

Question 1: What percent of the **right** circle do you think is covered by the shape in the **right** circle?

The percent of the circle covered is between 0% and 100%

☐ ☐

0 25 50 75 100

If the random number is below 0 and this question is used for payment:
You bet on the percent of the right circle covered by its shape. This means that you win the $30 prize with a chance that is equal to the percent of the right circle that is covered by its shape.

If the random number is above 100 and this question is used for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 0 and 100 and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or on the percent of the right circle covered by its shape.
If the coin comes up heads you bet on the percent of the right circle covered by its shape: You win the $30 prize with a chance that is equal to the percent of the right circle that is covered by its shape.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Question 2: What percent of the **left** circle do you think is covered by the shape in the **left** circle?

The percent of the circle covered is between 0% and 100%

☐ ☐

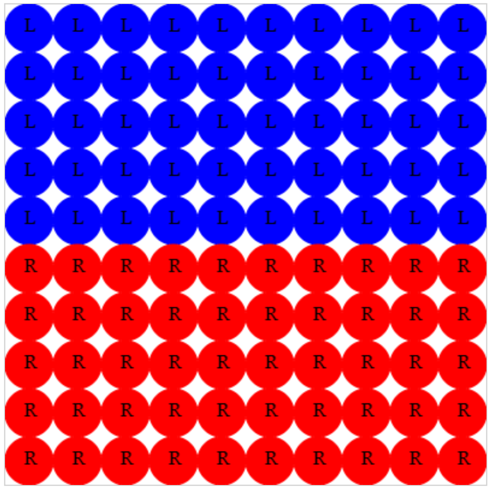
0 25 50 75 100

If the random number is below 0 and this question is used for payment:
You bet on the percent of the left circle covered by its shape. This means that you win the $30 prize with a chance that is equal to the percent of the left circle that is covered by its shape.

If the random number is above 100 and this question is used for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 0 and 100 and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or on the percent of the left circle covered by its shape.
If the coin comes up heads you bet on the percent of the left circle covered by its shape: You win the $30 prize with a chance that is equal to the percent of the left circle that is covered by its shape.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Question 3: Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options?



I would like to assign 50 balls to betting on the shape in the right circle

and I would like to assign 50 balls to betting on the shape in the left circle

☐

If this question is used for payment:

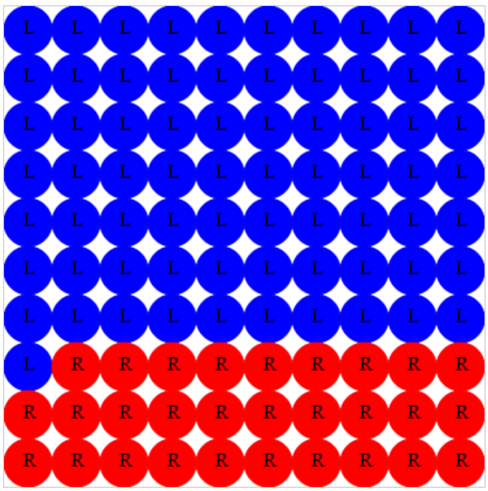
You have a 50% chance of betting on the shape in the right circle, in which case you win the $30 prize with a chance that is equal to the percent of the right circle that is covered by the shape in the right circle.

You have a 50% chance of betting on the shape in the left circle, in which case you win the $30 prize with a chance that is equal to the percent of the left circle that is covered by the shape in the left circle.

Next

Quiz time! Quiz 1, Question 3

Question 3: Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options?



I would like to assign 29 balls to betting on the shape in the right circle

and I would like to assign 71 balls to betting on the shape in the left circle

☐

If this question is used for payment:

You have a 29% chance of betting on the shape in the right circle, in which case you win the $30 prize with a chance that is equal to the percent of the right circle that is covered by the shape in the right circle.

You have a 71% chance of betting on the shape in the left circle, in which case you win the $30 prize with a chance that is equal to the percent of the left circle that is covered by the shape in the left circle.

If a Question 3 from the first 12 rounds like the one above is used for payment, which of the following is true about your chance of winning the $30 prize?

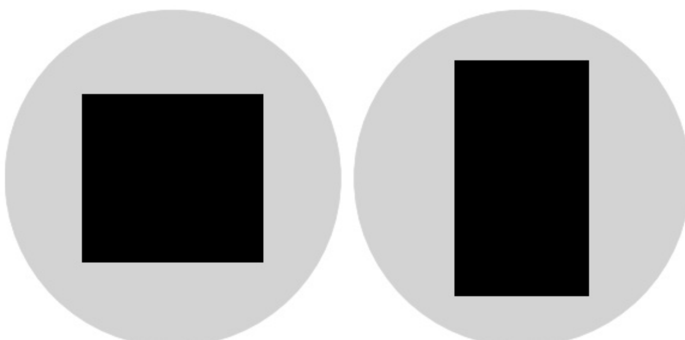
Next

Quiz time! Quiz 1, Question 4

Please look at the image of shapes below and answer the questions about your confidence in which is larger:

Remember: you must spend at least 45 seconds on this page. The "Next" button will not appear until 45 seconds pass.

This round is for practice only. However, you will be rewarded for correct responses to the quiz questions on the following pages.



Question 1: What percent of the **right** circle do you think is covered by the shape in the **right** circle?

The percent of the circle covered is between 0% and 100%

☐ ☐

0 25 50 75 100

If the random number is below 0 and this question is used for payment:
You bet on the percent of the right circle covered by its shape. This means that you win the $30 prize with a chance that is equal to the percent of the right circle that is covered by its shape.

If the random number is above 100 and this question is used for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 0 and 100 and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or on the percent of the right circle covered by its shape.

If the coin comes up heads you bet on the percent of the right circle covered by its shape: You win the $30 prize with a chance that is equal to the percent of the right circle that is covered by its shape.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Question 2: What percent of the **left** circle do you think is covered by the shape in the **left** circle?

The percent of the circle covered is between 0% and 100%

☐ ☐

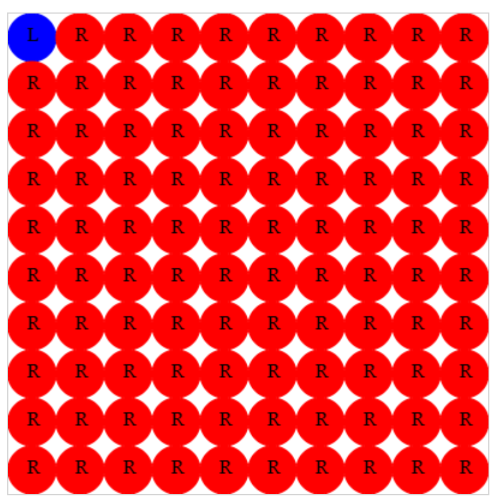
0 25 50 75 100

If the random number is below 0 and this question is used for payment:
You bet on the percent of the left circle covered by its shape. This means that you win the $30 prize with a chance that is equal to the percent of the left circle that is covered by its shape.

If the random number is above 100 and this question is used for payment:
You bet on the random number. This means that you win the $30 prize with a percent chance that is equal to the random number.

If the random number is equal to or between 0 and 100 and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or on the percent of the left circle covered by its shape.
If the coin comes up heads you bet on the percent of the left circle covered by its shape: You win the $30 prize with a chance that is equal to the percent of the left circle that is covered by its shape.
If the coin comes up tails you bet on the random number: you win the $30 prize with a percent chance that is equal to the random number.

Question 3: Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options?



I would like to assign 99 balls to betting on the shape in the right circle

and I would like to assign 1 balls to betting on the shape in the left circle

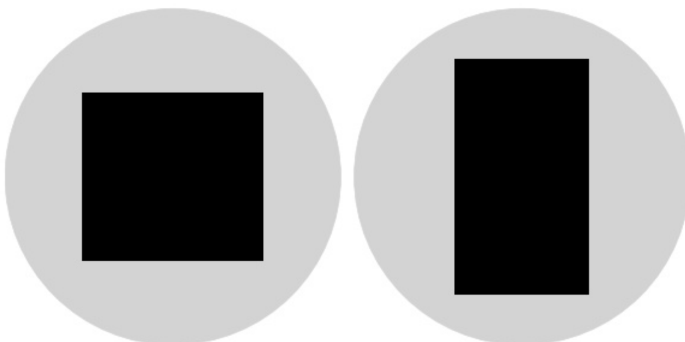
☐

If this question is used for payment:

You have a 99% chance of betting on the shape in the right circle, in which case you win the $30 prize with a chance that is equal to the percent of the right circle that is covered by the shape in the right circle.

You have a 1% chance of betting on the shape in the left circle, in which case you win the $30 prize with a chance that is equal to the percent of the left circle that is covered by the shape in the left circle.

Quiz time! Quiz 1, Question 4



Previously, you selected the following in response to the question: "What percent of the **left** circle do you think is covered by the shape in the **left** circle?"

The percent of the circle covered is between 47% and 50%

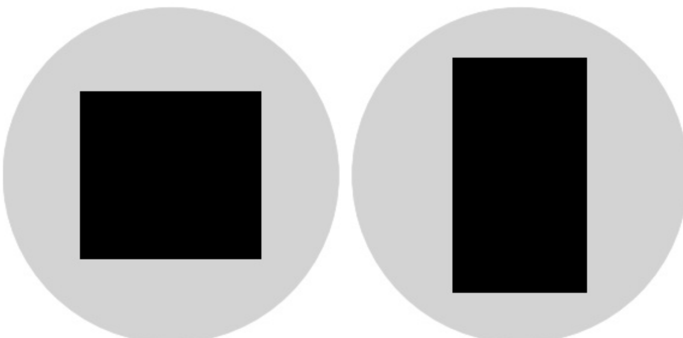
0 25 50 75 100

Suppose a random number is selected which is below 47, for example 15, and this question is used for payment:
You bet on the percent of the left circle covered by its shape. This means that you win the $30 prize with a chance that is equal to the percent of the left circle that is covered by its shape.

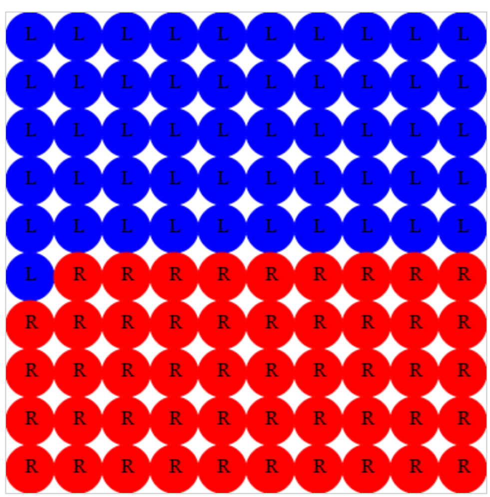
On the other hand, if the random number is above 50, for example 59, and this question is used for payment, which of the following statements is true?

Finally, if the random number is equal to or between 47 and 50, for example 49, and this question is used for payment:
A fair digital coin is flipped for you that determines if you bet on the random number or on the percent of the left circle covered by its shape.
If the coin comes up heads you bet on the percent of the left circle covered by its shape: you win the $30 prize with a chance that is equal to the percent of the left circle that is covered by its shape.
If the coin comes up tails you bet on the random number: you win the $30 prize with a 49 percent chance, a chance equal to the random number.

Quiz time! Quiz 1, Question 5



Previously, you selected the following in response to the question: "Would you rather bet on the shape in the right circle, the shape in the left circle, or randomize over the two options?"



I would like to assign 49 balls to betting on the shape in the right circle

and I would like to assign 51 balls to betting on the shape in the left circle

If this question is used for payment:

You have a 49% chance of betting on the shape in the right circle, as this is the chance that a red ball is randomly drawn. If you bet on the shape in the right circle, what would be the chance that you win the $30 prize?

A chance that is equal to the percent of the right

You have a 51% chance of betting on the shape in the left circle, as this is the chance that a blue ball is randomly drawn. If you bet on the shape in the left circle, you would win the $30 prize with a chance that is equal to the percent of the left circle that is covered by the shape in the left circle.

Next